m_s from sum rules

Consider the 2-point function of mesonic currents:

(Shifman, Vainshtein, Zakharov 1979)

$$\Pi(q^2) = i\int dx\, e^{iqx}\, \langle\Omega|\, T\{\, j(x)\, j(0)^\dagger\}\, |\Omega\rangle$$

with the current $j(x) = (\bar{q}^A \Gamma q^B)(x)$.

$\Pi(s{=}q^2)$ can be calculated within the Operator Product Expansion (OPE) in the deep Euclidean region. $(s \to -\infty)$

Operator Product Expansion:

$$\Pi_{\rm th}(s) = C_{\rm pt}(s) + \sum_i C_i(s)\langle\Omega|O_i(0)|\Omega\rangle$$

Also condensate corrections like $\langle m_q\bar{q}q\rangle$, $\langle\alpha_s FF\rangle$ or higher arise, being suppressed by inverse powers of s.

For the m_s determination, the divergence of vector or axialvector current is the most promising choice for $j(x)$:

$$j(x) = \partial^\mu(\bar{s}\gamma_\mu(\gamma_5)u)(x) = i\,(m_s \mp m_u)(\bar{s}(\gamma_5)u)(x)$$

It follows that:

$$\Pi(s) = (m_s \mp m_u)^2\,\Pi^{S,P}(s)$$

Thus the 2-point function $\Pi(s)$ has a global m_s^2 dependence.

Another possibility are QCD sum rules with vector or axialvector correlators like hadronic τ decays or $R_{e^+e^-}(s)$.

However, here m_s only appears in power corrections like m_s^2/M_τ^2 or m_s^2/s, and the perturbative contribution needs to be subtracted to obtain a sufficient sensitivity to m_s.

Dispersion relation

A relation between $\Pi(s)$ in the Euclidean and the physical regions can be obtained from its analyticity properties.

Dispersion relation:

$$\Pi(s) \;=\; \int\limits_0^\infty \frac{\rho(s')}{(s'-s-i\epsilon)}\, ds' + \text{subtract.}$$

$\rho(s) = 1/\pi\,\mathrm{Im}\,\Pi(s)$ is the spectral function.

Integral transforms are usually applied in order to enhance the energy region of interest. $\Rightarrow$ E.g. Borel sum rules.

$$u\,\widehat{\Pi}_{\rm th}(u) \;=\; \int\limits_0^{s_0} ds\, e^{-s/u} \rho_{\rm ph}(s) + \int\limits_{s_0}^\infty ds\, e^{-s/u} \rho_{\rm th}(s)\,.$$

Scalar sum rule

The perturbative contribution is known analytically up to $\mathcal{O}(\alpha_s^4)$:

(Baikov, Chetyrkin, Kühn 2005)

$$\widehat{\Pi}_{\rm pt}(u) = \frac{N_c}{8\pi^2} m_s^2(u) u \Big[1 + 1.5\alpha_s + 2.2\alpha_s^2 + 1.7\alpha_s^3 - 0.32\alpha_s^4 \Big]$$

Even for $\alpha_s(1) \approx 0.5$, series displays reasonable convergence.

Power corrections:

Already at $u = 1$ GeV^2, the power corrections are small:

$$\mathcal{O}\left(\frac{m_s^2}{u}\right) \approx 2\,\%\,, \qquad \mathcal{O}\left(\frac{\langle O_4\rangle}{u^2}\right) < 1\,\%$$

The are included in the phenomenological analysis.

Hadronic spectral function

Generally, the hadronic spectral function is given by:

$$\rho_{\rm ph}(s) = (2\pi)^3 \sum\!\!\!\!\!\!\!\int_\Gamma \langle j(0)|\Gamma\rangle\langle\Gamma|j(0)^\dagger\rangle\, \delta^{(4)}(q-p_\Gamma)\,.$$

Scalar case:

(Narison, Paver, de Rafael, Treleani 1983)

Lowest lying state: $(K\pi)$-system in S-wave with $I=1/2$.

$$\rho_{\rm ph}(s) \geq \frac{3\Delta_{K\pi}^2}{32\pi^2} \sum_i \sigma_i(s)\,|F_i(s)|^2$$

with $i = K\pi,\ K\eta,\ K\eta'$ and $\Delta_{K\pi} = M_K^2 - M_\pi^2$.

Phase-space factor $\sigma_{K\pi}(s) = \sqrt{(1-s_+/s)(1-s_-/s)}$ with

$$s_+ = (M_K+M_\pi)^2\,, \quad s_- = (M_K-M_\pi)^2$$

Hadronic spectral function

The corresponding scalar resonances which contribute in this system are the $K_0^*(800)$, the $K_0^*(1430)$, and the $K_0^*(1950)$.

Including these states should give a good description for the hadronic spectral function up to about 2 GeV.

Finally, the scalar form factors are defined by:

$$i\,\langle\Omega|\,\partial^\mu(\bar{s}\gamma_\mu u)\,|\Gamma\rangle = \frac{\Delta_{K\pi}}{\sqrt{2}}\,C_\Gamma\,F_\Gamma(s)$$

where the C_Γ are Clebsch-Gordan coefficients.

As yet, the form factors $F_i(s)$ have not been measured directly. Thus an indirect determination is required.

Hadronic spectral function

From unitarity we have the following relation:

$$\mathrm{Im} F_k(s) = \sum_i \sigma_i(s)\, F_i(s)\, t_0^{ik}(s)^*$$

with $t_0^{ik}(s)$: S-wave $I=1/2$ scattering amplitudes.

The $F_i(s)$ also satisfy dispersion relations. In the 2-channel case with $F_1 \equiv F_{K\pi}$ and $F_3 \equiv F_{K\eta'}$:

$$F_1(s) = \frac{1}{\pi}\int_{s_1}^{\infty} \frac{\sigma_1 F_1\, t_0^{11}(s')^*}{(s'-s-i0)}\, ds' + \frac{1}{\pi}\int_{s_3}^{\infty} \frac{\sigma_3 F_3\, t_0^{13}(s')^*}{(s'-s-i0)}\, ds'$$

$$F_3(s) = \frac{1}{\pi}\int_{s_1}^{\infty} \frac{\sigma_1 F_1\, t_0^{13}(s')^*}{(s'-s-i0)}\, ds' + \frac{1}{\pi}\int_{s_3}^{\infty} \frac{\sigma_3 F_3\, t_0^{33}(s')^*}{(s'-s-i0)}\, ds'$$

Numerical analysis

Several ingredients are required for solving the set of coupled integral equations:

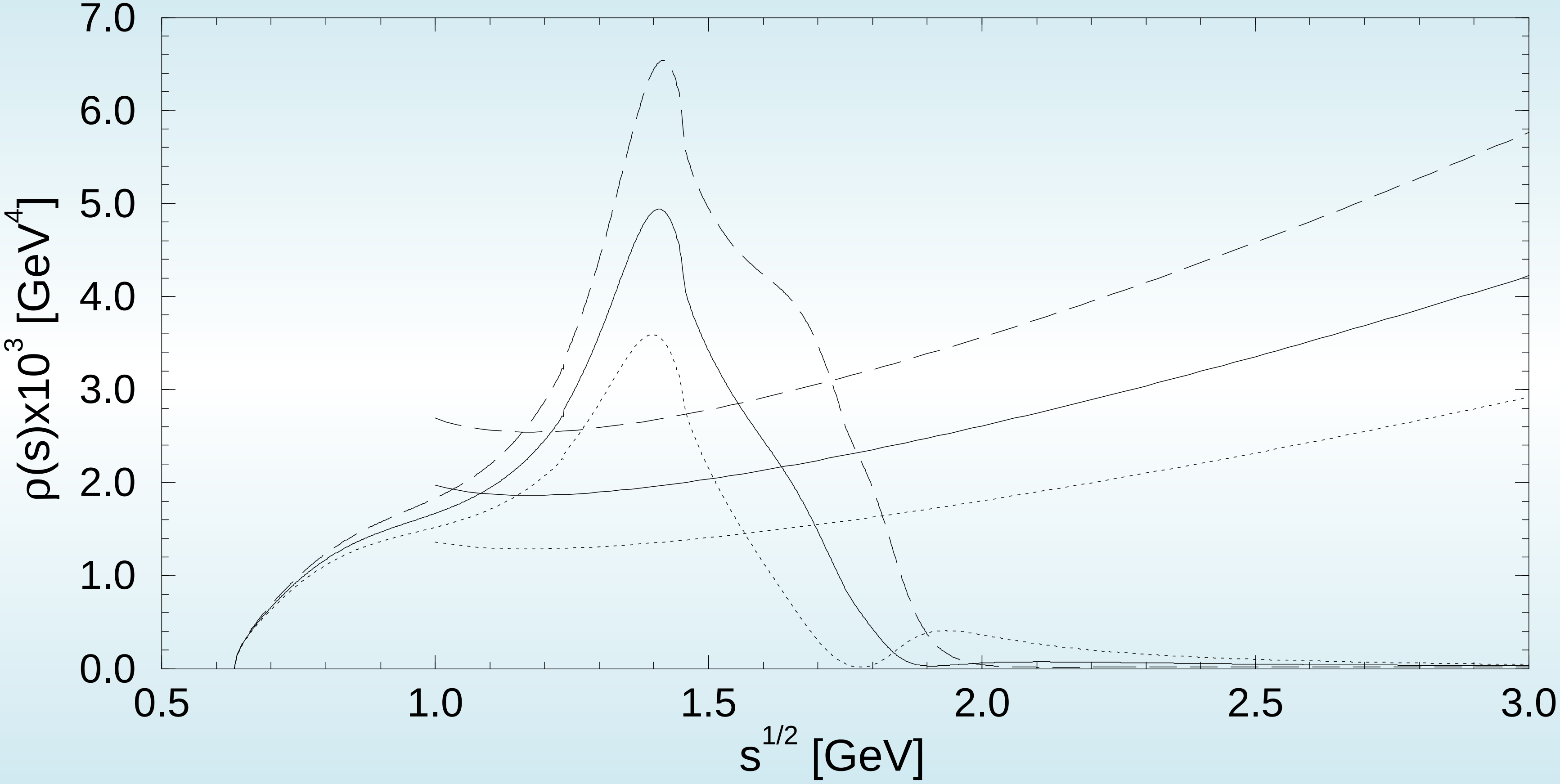
- An input for the scattering amplitudes is obtained by fitting an Ansatz from resonance ChPT to experimental data for S-wave $K\pi$ scattering. (M.J., Oller, Pich 2000/02)

- Two integration constants are also required. These can be chosen to be: $F_{K\pi}(0) = 0.972 \pm 0.012$ and

$$\frac{F_{K\pi}(\Delta_{K\pi})}{F_{K\pi}(0)} = \frac{F_K}{F_\pi F_{K\pi}(0)} + \frac{\Delta_{\rm CT}}{F_{K\pi}(0)} = 1.2346(53)$$

- The last relation follows from the ratio of leptonic K and π decays, as well as $|V_{us}|F_{K\pi}(0)$ from K_{l3} decays.

Hadronic spectral function



Numerical analysis

Putting together the theoretical 2-point function and the hadronic spectral function, the strange mass is found to be:

(M.J., Oller, Pich 2002/06)

$$\Rightarrow \qquad m_s(2\,\text{GeV}) = 87.6\,{}^{+\,8.8}_{-\,6.8}\ \text{MeV}$$

The dominant uncertainty $({}^{+\,7}_{-\,5}\ \text{MeV})$ resides in the hadronic spectral function, in particular in the value for $F_{K\pi}(\Delta_{K\pi})$.

About 3 MeV uncertainty are due to the input value for α_s, and the matching scale of the hadronic spectral function and the perturbative continuum $\sqrt{s_0} = (2.0 - 2.4)$, each.

All remaining error contributions are about 1 GeV or smaller.

Hadronic τ decay rate

Consider the physical quantity R_τ: (Braaten, Narison, Pich (1992))

$$R_\tau \equiv \frac{\Gamma(\tau^- \to \text{hadrons}\,\nu_\tau(\gamma))}{\Gamma(\tau^- \to e^-\bar{\nu}_e\nu_\tau(\gamma))} = 3.642 \pm 0.012\,.$$

R_τ is related to the QCD correlators $\Pi^{T,L}(z)$: $(z \equiv s/M_\tau^2)$

$$R_\tau = 12\pi \int\limits_0^1 dz(1-z)^2\Big[(1+2z)\,\mathrm{Im}\Pi^T(z) + \mathrm{Im}\Pi^L(z)\Big]\,,$$

with the appropriate combinations

$$\Pi^J(z) = |V_{ud}|^2\Big[\Pi_{ud}^{V,J} + \Pi_{ud}^{A,J}\Big] + |V_{us}|^2\Big[\Pi_{us}^{V,J} + \Pi_{us}^{A,J}\Big]\,.$$

Additional information can be inferred from the moments

$$R_\tau^{kl} \equiv \int_0^1 dz\,(1-z)^k z^l \,\frac{dR_\tau}{dz} = R_{\tau,NS}^{kl} + R_{\tau,S}^{kl}\,.$$

Theoretically, R_τ^{kl} can be expressed as:

$$R_\tau^{kl} = N_c\, S_{\rm EW}\Big\{(|V_{ud}|^2+|V_{us}|^2)\Big[1+\delta^{kl(0)}\Big]$$

$$+\sum_{D\geq 2}\Big[|V_{ud}|^2\delta_{ud}^{kl(D)}+|V_{us}|^2\delta_{us}^{kl(D)}\Big]\Big\}\,.$$

$\delta_{ud}^{kl(D)}$ and $\delta_{us}^{kl(D)}$ are corrections in the Operator Product Expansion, the most important ones being $\sim m_s^2$ and $m_s\langle\bar{q}q\rangle$.

The sensitivity to the strange quark mass can be enhanced by considering the flavour SU(3)-breaking difference:

(Pich, Prades; ALEPH (1998))

$$\delta R_\tau^{kl} \equiv \frac{R_{\tau,NS}^{kl}}{|V_{ud}|^2} - \frac{R_{\tau,S}^{kl}}{|V_{us}|^2} = 3\, S_{\rm EW} \sum_{D\geq 2} \Big(\delta_{ud}^{kl(D)} - \delta_{us}^{kl(D)} \Big).$$

Flavour independent uncertainties drop out in the difference.

In previous analyses a sizeable part of the theoretical error was due to large α_s corrections in the longitudinal contribution.

This uncertainty could be greatly reduced by replacing badly behaved scalar/pseudoscalar correlators with phenomenology.

(Gámiz, M.J., Pich, Prades, Schwab 2003/05)

Strange mass from R_τ

Taking a weighted average of the strange mass extractions, for the $(2,0)$ to $(4,0)$ moments, we obtain

(Gámiz, M.J., Pich, Prades, Schwab 2003/05)

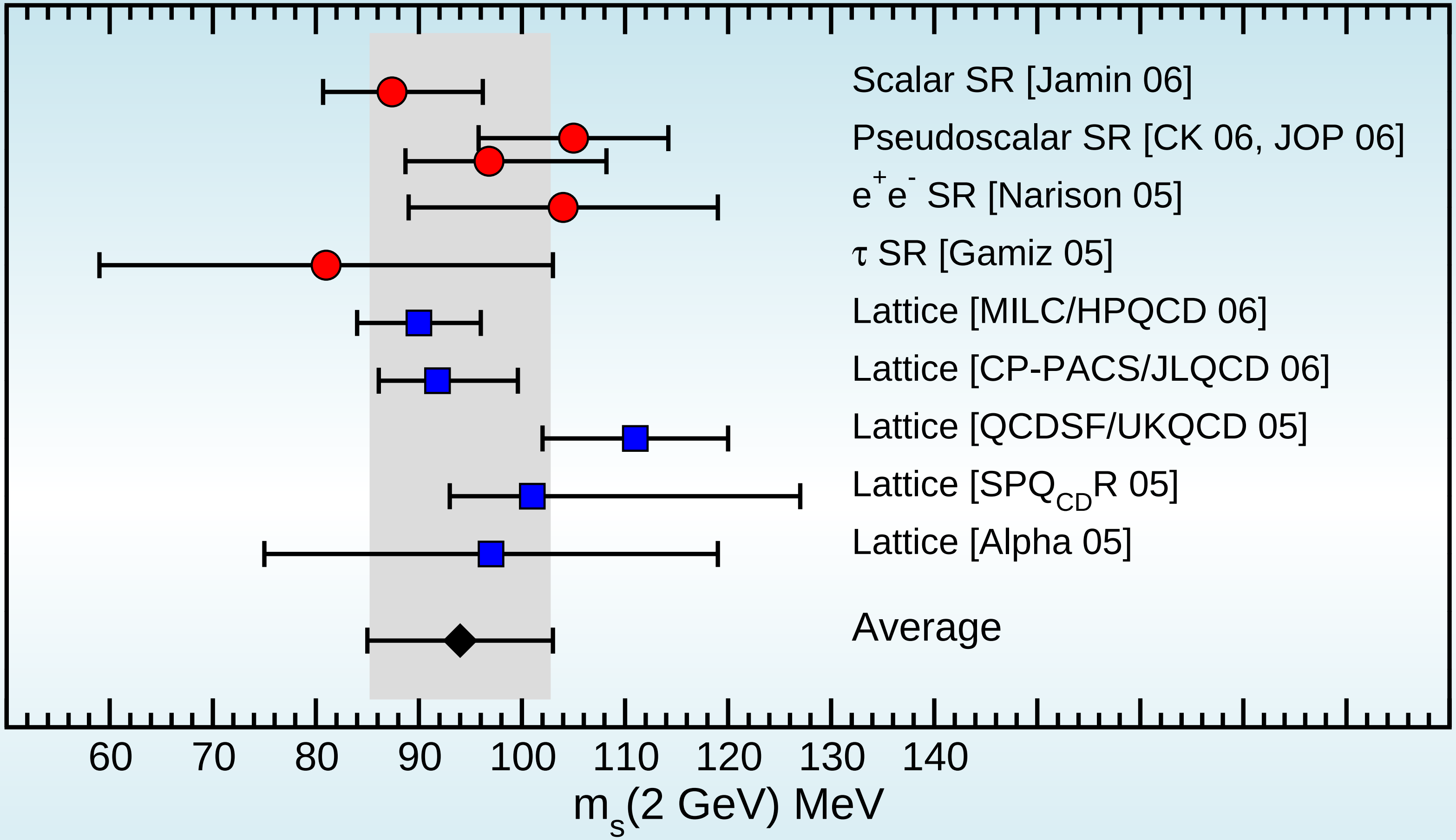
$$m_s(M_\tau) = 84 \pm 23 \text{ MeV} \Rightarrow \boxed{m_s(2\,\text{GeV}) = 81 \pm 22 \text{ MeV}}$$

The dominant error contributions are due to the experimental moments, V_{us}, and higher-order perturbative corrections.

The strong k-dependence, of m_s present in the analysis of the ALEPH data, is reduced for a world-av. spectral function.

Performing a simultaneous fit for V_{us} and m_s to the first five $(k,0)$ moments, we find: $|V_{us}| = 0.2196$ and $m_s = 76$ MeV.

Strange quark mass



$\Rightarrow$ Average: $m_s(2\,\text{GeV}) = 94 \pm 9\,(6)$ MeV

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