Status of Higher Order Calculations of R_{had} and determimation of α_s in e^+e^- annihilation

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Phys. Rev. Lett. 88 (2002) 012001

Phys. Rev. D67 (2003) 074026

Phys. Letters. B559 (2003) 245

Nucl. Phys. Proc. Suppl. 135 (2004) 243-246

Phys. Rev. Lett. 95, 012003 (2005)

$+$ few manuscripts in preparation

MINI WORKSHOP des SFB/TR9, Aachen, 21 Februar 2007

- Intro: α_s from $R(s)$ and τ decays
- $R(s)$: Theoretical Framework: the Adler Function and All That
- kinematical versus dynamical contributions to the Adler function and $R(s)$
- vector and scalar correlators: current status and what is/not possible in foreseeable future
- Lessons for the α_s determinations from the τ decays

Precize and reliable determinations of $\alpha_s(Q)$ come mainly from

- e^+e^- colliders (LEP: inclusive hadronic Z decay, inclusive hadronic τ decay, event shapes and jet rates)

- scaling violations in Deep Inelastic Scattering (DIS)

In this talk we concentrate on the inclusive hadronic Z decays. For measuring $\alpha_s(Q^2)$ the basic quantity is Γ_h the Z hadronic partial width.

The corresponding theoretical object is the famous R-ratio:

$$\boldsymbol{R(s) = \sigma_{tot}(e^+e^- \to hadrons)/\sigma(e^+e^- \to \mu^+\mu^-)}$$

Z decay widths are theoretically remarkably clean:

- the PT corrections are known to ${\alpha_s}^3$ (and partially even to ${\alpha_s}^4$, see below)

- power corrections are small and well-undestood:

1. known to ${\alpha_s}^4 \, \frac{\bar{m}_b^2}{M_Z}$ /P. Baikov, K. Ch, J.Kühn, 2004/

2. "doubly" well-suppressed due to the small $\bar{m}^2(M_Z)_b/M_Z^2$ ratio (note that it is running mass of the $b-$quark enters here / K. Ch, J.Kühn, 1991/):

$$\bar{m}(M_Z) \approx 1/2 \, m_b^{pole} \approx 2.5 \, \text{GeV}$$

3. even term of order ${\alpha_s}^3 \, \frac{\bar{m}_b^4}{M_Z^2}$ is available

/K. Ch, R. Harlander, J.Kühn, 2000/

From measurements at Z-peak LEPEWWG arrives at:

$$\alpha_s(M_Z) = 0.1186(27)$$

with predominantly theoretical error from uncalculated

higher orders! α_s^4 (massless diagrams!)

theory error

smaller than **present** experimental error (but not much!)

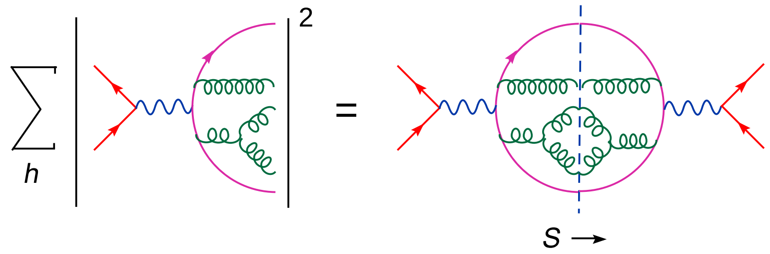
highly relevant for GIGA-Z

$\Longrightarrow$ calculation of α_s^4-term required for GIGA-Z

and even more so for $R_\tau = \Gamma(\tau \to \nu \text{ had})/\Gamma(\tau \to e\nu\nu)$

Theoretical Framework

R(s) is related (via unitarity) to the correlator of the EM quark currents:



$$R(s) \approx \Im\,\Pi(s - i\delta)$$

$$3Q^2\Pi(q^2 = -Q^2) = \int e^{iqx}\langle 0|T[\ j^v_\mu(x) j^v_\mu(0)\ ]|0\rangle dx$$

To conveniently sum the RG-logs one uses the Adler function:

$$D(Q^2) = Q^2 \frac{d}{d\,Q^2}\Pi(q^2) = Q^2 \int \frac{R(s)}{(s+Q^2)^2}\,ds$$

or $(a_s \equiv \alpha_s/\pi)$

$$R(s) = \frac{1}{2\pi i}\int_{-s-i\delta}^{-s+i\delta} dQ^2 \frac{D(Q^2)}{Q^2} = D(s) - \pi^2 \frac{\beta_0^2 d_0}{3} a_s^3 + \ldots$$

In the massless case the PT series look like

$$D = \sum_{i,j<l} d_{i,j}\, a_s(\mu)^i \ln\frac{\mu^2}{Q^2} \quad \text{and} \quad R = \sum_{i,j<l} r_{i,j}\, a_s(\mu)^i \ln\frac{\mu^2}{s}$$

as both D and R-functions are the physical RG-invariant ones, so one could conveniently sum RG logs of μ^2/Q^2 (in D) and μ^2/s (in R) by setting $\mu = Q$ and $\mu = s$ respectively:

$$D = \sum_{i\geq 1} d_i\, a_s(Q)^i \quad \text{and} \quad R = \sum_{i\geq 1} r_i\, a_s(s)^i$$

$r_0 = d_0 = 1$, $r_1 = d_1 = 1$, $r_2 = d_2 = 1.98 - 0.115\, n_f$, but already $r_3 \neq d_3$ due to the effects of the analytical continuation like

$$\ln^3(Q^2/\mu^2) \rightarrow [3\ln^2(s/\mu^2) - \pi^2]$$

The effect starts at a_s^3: $r_3 = d_3 - \pi^2\frac{\beta_0^2}{3}$ and is important numerically, say, for $n_f = 3$

$$r_3 = (d_3 = 6.37) - \boxed{16.64} = -10.27$$

On obvious reasons we will call these π^2 terms in R(s) as "kinematical ones" and those coming from d_i as "dynamical"

The hypothesis that the π^2 terms continue to be dominant in higher orders is sitting behind two important developments:

A. the commonly accepted estimation[1] from the PMS principle of r_4

$$r_4(n_f = 5) = -97$$

comes almost completely from π^2 terms

$$r_4(n_f = 5) = (d_4^{PMS} = 8) - 105$$

In fact, the quality of PMS-predictions for d_3 (where the exact result is known) is far from being good:

$$d_3^{exact}(n_f = 5) = -0.69, \quad d_3^{PMS}(n_f = 5) = 3.77$$

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[1] /Kataev and Starshenko, 1994/

B.Contour Improvement[1] PT (CIPT) versus Fixed Order PT (FOPT)

in extracting α_s from R_τ

$$R_\tau = \frac{\Gamma(\tau \to \nu_\tau \text{hadrons})}{\Gamma(\tau \to \nu_\tau e \bar{\nu}_e)} \sim \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + 2\frac{s}{M_\tau^2}\right) R(s)$$

PDG cites:

$$\alpha_s(M_Z) = .120(3)$$

as obtained from

$$\alpha_s(M_\tau) = 0.334 \pm 0.007_{\text{exp}} \pm 0.021_{\text{theo}}$$

and

$$\alpha_s(M_\tau) = 0.3478 \pm 0.009_{\text{exp}} \pm 0.019_{\text{theo}}$$

by ALEPH and OPAL collaborations repectively

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[1] A.A. Pivovarov (1991,1992); F. Le Diberder and A. Pich (1992)/

Attempts to soft the difference between CIPT and FOPT results[1]

Implication for α_s

with $\alpha_s^4 \to 0$

$$\alpha_s^{\mathrm{FOPT}}(M_\tau) = 0.345 \pm (0.025|0.037)$$
$$\alpha_s^{\mathrm{CIPT}}(M_\tau) = 0.364 \pm (0.012|0.021)$$
$$\alpha_s^{\mathrm{FOPT}}(M_Z) = 0.1209 \pm (0.0024|0.0037)$$
$$\alpha_s^{\mathrm{CIPT}}(M_Z) = 0.1229 \pm (0.0011|0.0020)$$

with α_s^4 and α_s^5

Method	$\alpha_s(M_\tau)$	$\Delta\, \delta_P^{\mathrm{exp}}$	$\Delta\, \mu$	$\Delta\, d_0^{[1]4}$	$\Delta\, d_0^{[1]5}$
FOPT	$0.330 \pm 0.006 \pm 0.02$	0.006	0.019	0.0045	0.0011
CIPT	$0.354 \pm 0.009 \pm 0.006$	0.009	0.0036	0.0042	0.0019

[1] /P. Baikov, K. Ch, J.Kühn, PRD 67, 074026 (2003)/

$$\Rightarrow \quad \alpha_s^{\mathrm{FOPT}}(M_Z) = 0.1192 \pm 0.0007 \pm 0.002$$

$$\alpha_s^{\mathrm{CIPT}}(M_Z) = 0.1219 \pm 0.001 \pm 0.0006$$

uncertainty is reduced; difference between FOPT and CIPT remains!

[this difference is reduced for a fictitious heavy lepton of 3 GeV]

Tool Box

- reduction to Masters: "direct and automatic" construction of CF's through $1/D$ expansion—made with BAICER—within the Baikov's representation for Feynman integrals[1]
- all 4-loop master p-integrals are all known analytically /P. Baikov and K.Ch. (2004)/
- computing time and required resources: could be huge (the price for full automatization); to cope with it we use parallel FORM /Vermaseren, Retey, Fliegner, Tentyukov, ...(2000 – . . .)

[1] Baikov, Phys. Lett. B385 (1996) 403; B474 (2000) 385; Nucl.Phys.Proc.Suppl.116:378-381,2003

CURRENT STATUS OF THE TECHNOLOGY

BEFORE (1990's): 3-loop propagators (including real parts, MINCER) and the absorpive parts of 4-loop ones can be *analytically* computed in any massless theory $\Longrightarrow$ correlators to $\mathcal{O}(\alpha_s^3)$ /**4**-loops/

NOW: 4-loop propagators (including real parts) and the absorpive parts of 5-loop ones can be *analytically* computed in any massless theory

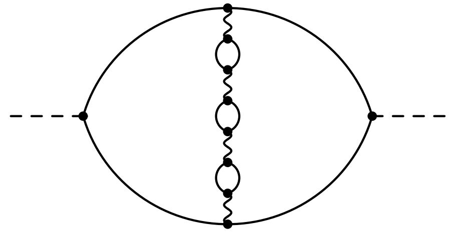
$\Longrightarrow$ correlators to $\mathcal{O}(\alpha_s^3)$ /**5**-loops/

FUTURE: 5-loop (real) + 6 -loop (absorptive) $\Longrightarrow$ **NOPE**

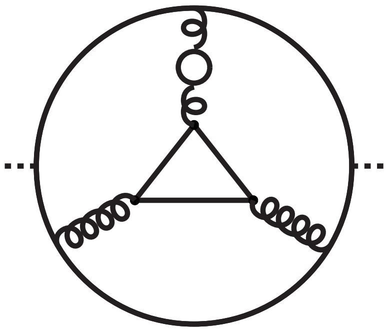
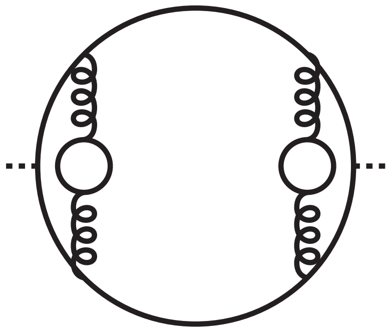
Status: 5-loop VV-correlator (massless case):

leading and subleading n_f terms for $R_{e^+e^-}$, R_τ

$\alpha_s^4 n_f^3$ (renormalon chain, known since long /M. Beneke, 1993/):



$\alpha_s^4 n_f^2$ (since 2002)

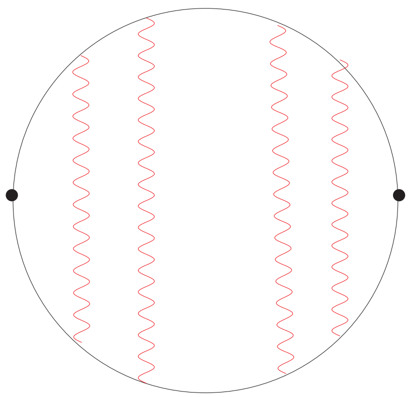
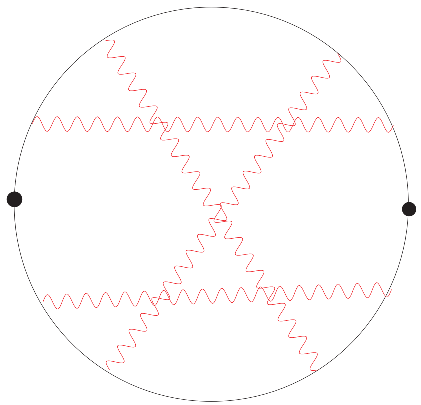
$+$   $+ \approx 100$ more

$$\mathbf{R} = \mathbf{1} + \mathbf{a_s} + \mathbf{a_s^2}\,(\mathbf{1.99} - \mathbf{0.115\ n_f}) \ + a_s^3 \left(-6.64 - 1.2\ n_f - 0.0053\ n_f^2\right)$$

$$+\mathbf{a_s^4}\left(\mathbf{0.0215\ n_f^3} - \mathbf{0.797\ n_f^2} + \ldots\right) \qquad (\mathbf{a_s} \equiv \alpha_\mathbf{s}(\mathbf{s})/\pi)$$

Note that: $\quad -0.797 = 1.876^D - 2.673^{\pi^2}$!!!

very recently we have computed the purely abelian, quenched ($\mathcal{O}(C_F^4)$) term in R(s):

 $+$  $+$ 2671 diagrams more

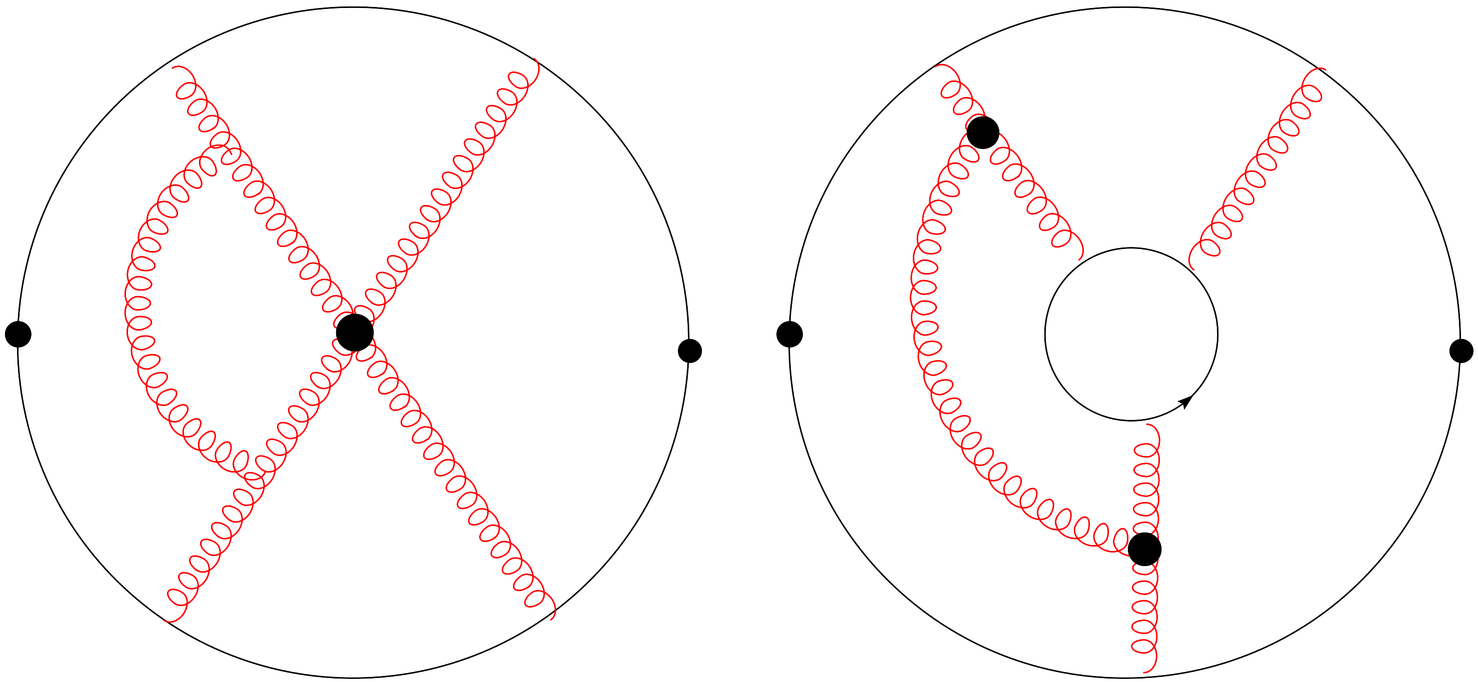
Theoretically the calculation is important as

- it comprises all vector topologies and is very comparable in complexity with the full calculation
- it has interesting impilications for the long standing problem of rationally of the quenched QED β function

Unfortunately, the term is numerically rather well-suppressed in R(s):

$\approx -8(\alpha_s/\pi)^4$ /cmp. to a typical PMS prediction (Kataev, 1994): $-97(\alpha_s/\pi)^4$/

We should wait till the end of (currently running) calculation of the remaining $\approx 17 \cdot 10^3$ nonabelian or/and non-quenched diagrams like



Recent Results* related to R(s)

- the first complete five-loop results in QCD and QED:

$\mathcal{O}(\alpha_s{}^4 m_q^2/s)$ contribution to $R(s)$

R^{SS} $(\sim \Gamma(H \longrightarrow \text{hadrons}))$ to order $\mathcal{O}(\alpha_s{}^4)$

QED β-function in quenched approximation

$(\equiv C_F^4 \alpha_s{}^4$ piece in $R(s))$

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* P. Baikov, K. Ch, J.Kühn, 2004 – 2006

Scalar Correlator in 5 loops

$\widetilde{R}(s) = \mathrm{Im}\,\widetilde{\Pi}(-s - i\epsilon)/(2\pi\, s)$ is the absorptive part of the scalar two-point correlator:

$$\widetilde{\Pi}(Q^2) = (4\pi)^2 i \int dx e^{iqx} \langle 0|\, T[\, J_f^{\mathrm{S}}(x) J_f^{\mathrm{S}}(0)\,]\, |0\rangle$$

Again consider Adler function:

$$\widetilde{D}(Q^2) = \frac{Q^2}{6} \frac{\mathrm{d}}{\mathrm{d}Q^2} \frac{\widetilde{\Pi}(Q^2)}{Q^2} = \int_0^\infty \frac{Q^2\ \widetilde{R}(s) ds}{(s+Q^2)^2}\ ,$$

$$\widetilde{D}(Q^2) = 1 + \sum_{i=1}^{\infty}\ \tilde{d}_i a_s^i(Q^2), \quad \tilde{R}(s) = 1 + \sum_{i=1}^{\infty}\ \tilde{r}_i a_s^i(s)\ ,$$

$$
\begin{aligned}
d_4 = \;& n_f^3 \left[-\frac{520771}{559872} + \frac{65}{432}\zeta_3 + \frac{1}{144}\zeta_4 + \frac{5}{18}\zeta_5\right] \\
+ \;& n_f^2 \left[\frac{220313525}{2239488} - \frac{11875}{432}\zeta_3 + \frac{5}{6}\zeta_3^2 + \frac{25}{96}\zeta_4 - \frac{5015}{432}\zeta_5\right] \\
+ \;& n_f \left[-\frac{10458211915}{373248} + \frac{5747185}{5184}\zeta_3 - \frac{955}{16}\zeta_3^2 - \frac{9131}{576}\zeta_4\right. \\
+ \;& \left. \frac{41215}{432}\zeta_5 + \frac{2875}{288}\zeta_6 + \frac{665}{72}\zeta_7\right] \\
+ \;& \left[\frac{10811054729}{497664} - \frac{3887351}{324}\zeta_3 + \frac{458425}{432}\zeta_3^2\right. \\
+ \;& \left. + \frac{265}{18}\zeta_4 + \frac{373975}{432}\zeta_5 - \frac{1375}{32}\zeta_6 - \frac{178045}{768}\zeta_7\right]
\end{aligned}
$$

the resulting $\widetilde{R}$ reads

$$\begin{aligned}\widetilde{R} &= 1 + 5.6667 a_s + [35.94 - 1.359\, n_f]\, a_s^2 \\ &+ a_s^3 \left[164.14 - 25.77\, n_f + 0.259\, n_f^2\right] \\ &+ a_s^4 \left[39.34 - 220.9\, n_f + 9.685\, n_f^2 - 0.0205\, n_f^3\right] . \end{aligned} \quad (1)$$

and with "kinematical" π^2 terms explicitly separated and underlined:

$$\begin{aligned}\widetilde{R} = 1 + 5.667 a_s + a_s^2 \left[51.57 - \underline{15.63} - n_f(1.907 - \underline{0.548})\right] \\ + a_s^3 \left[648.7 - \underline{484.6} - n_f(63.74 - \underline{37.97}) + n_f^2(0.929 - \underline{0.67})\right] \\ + a_s^4 \left[9471. - \underline{9431.} - n_f(1454.3 - \underline{1233.4}) + n_f^2(54.78 - \underline{45.10}) \right. \\ \left. - n_f^3(0.454 - \underline{0.433})\right] \end{aligned}$$

remarkable mutual cancellations in all n_f powers!!!

for $n_f = 3 \longrightarrow a_s^4(5589 - 6126) = -536.8$

similar cancellations happen for ${\alpha_s}^4 m_a^2/s$ and ${\alpha_s}^4 n_f^2$ terms in R(s)

Comparison to PMS[1] |APAP[2] |NNA[3] predictions

Exact: $d_4 = 5588.7$ and $r_4 = -536.84$

$$d_4(\text{FAC/PMS}) = 5180 \longrightarrow r_4(\text{FAC/PMS}) = -945.28$$

$$r_4(\text{direct application of FAC/PMS in Minkowskian region}) = -528$$

$$d_4(\text{APM}) = 6214 \longleftarrow r_4(\text{APM}) = 195$$

$$d_4(\text{NNA}) = 1116 \longrightarrow r_4(\text{NNA}) = -5009$$

[1] Principle of Minimal Sensitivity (PMS): K.Ch., Kniehl, Sirlin, PRB 402 (1997) 359

[2] Asymptotic Padé-Approximant Method (APAM): Chishtie, Elias, Steele, PRD 59 (1999) 105013

[3] ‘Naive NoNabelization (NNA): Grosin,Broadhurst, PRD 52 (1995) 4082

Summary

- important partial results on $R(s)$ at α_s^4 order are already available
- full $\mathcal{O}({\alpha_s}^4)$ calculation is under way
- higher order terms in massless correlators display remarkable cancellations between kinematical and dynamical contributions
- as a result:

  1.it might be dangerous to assume that an account of kinematical π^2 terms in higher orders will produce a better approximation to the exact result

  2. in particularly, the use of contour improved perturbation theory for the τ-lepton decays seems to be disfavored