Heavy Quark Contributions to DIS

Isabella Bierenbaum

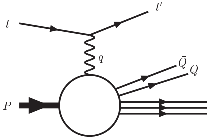
DESY, Zeuthen

21.2.2007

in collaboration with J. Blümlein and S. Klein

- I. Bierenbaum, J. Blümlein and S. Klein, Nucl. Phys. Proc. Suppl. **160**, (2006).
- J. Blümlein, A. De Freitas, W. L. van Neerven and S. Klein, Nucl. Phys. B **755** (2006).

Introduction



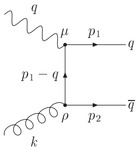
Kinematic variables:

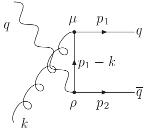
$$Q^2 := -q^2 \quad \nu := \frac{Pq}{M}, \quad x := \frac{Q^2}{2Pq} \quad \text{Bjørken-x}$$

Consider single photon exchange

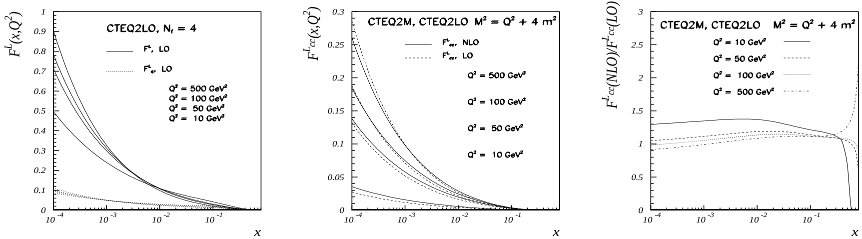
$$W_{\mu\nu}^{Q\overline{Q}}(q,P) = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P\rangle_{Q\overline{Q}}$$
$$= \frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L^{Q\overline{Q}}(x,Q^2)$$
$$+ \left(-\frac{Pq}{Q^2}g_{\mu\nu} + \frac{(q_\mu P_\nu + q_\nu P_\mu)}{Q^2} + \frac{P_\mu P_\nu}{Pq}\right) F_2^{Q\overline{Q}}(x,Q^2)$$

LO contribution to $F_{2,L}^{Q\overline{Q}}(x,Q^2)$:





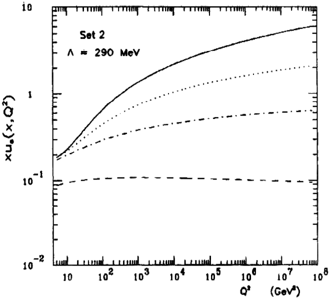
⇝ Heavy Flavour contributions to DIS structure functions are rather large. Example: $F_L(x, Q^2)$



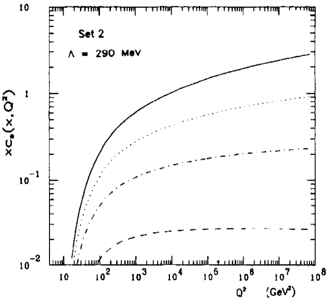
→ Different scaling violations for massive and massless terms

→ HFC currently known at NLO for unpolarized DIS:
E. Laenen et al., Nucl. Phys. B 392 (1993) 162, 229;
S. Riemersma et al., Phys. Lett. B **347** (1995) 143

→ Feasible way to $O(\alpha_S^3)$ results → results for $F_L^{Q\overline{Q}}(x, Q^2)$ derived by:
J. Blümlein et al., Nucl. Phys. B **755** (2006) 272



up-sea



charm-sea

Eichten et al., 1984

The Method

In Bjørken limit, $\{Q^2, \nu\} \to \infty$, x fixed: terms with $\xi^2 \approx 0$ dominate

$\Rightarrow$ OPE in light-cone expansion:

$$\lim_{\xi^2 \to 0} \Big[J(\xi), J(0) \Big] \propto \sum_{i,N,\tau} c^N_{i,\tau}(\xi^2, \mu^2) \xi_{\mu_1} ... \xi_{\mu_N} O^{\mu_1 \cdots \mu_m}_{i,\tau}(0, \mu^2)$$

Twist-2 operators (leading):

Flavour singlet operators:

$$\begin{aligned} O_q^{\mu_1, \dots, \mu_N}(z) &= \frac{1}{2} i^{N-1} S[\overline{q}(x)\gamma^{\mu_1} D^{\mu_2} \dots D^{\mu_N} q(x)] - \text{TraceTerms} \,, \\ O_g^{\mu_1, \dots, \mu_N}(z) &= \frac{1}{2} i^{N-2} S[F_\alpha^{a,\mu_1}(z) D^{\mu_2} \dots D^{\mu_{N-1}} F^{a,\alpha,\mu_N}(z)] - \text{TraceTerms} \,, \end{aligned}$$

Flavour non-singlet operator:

$$O_{q,r}^{\mu_1, \dots, \mu_N}(z) = \frac{1}{2} i^{N-1} S[\overline{q}(x)\gamma^{\mu_1} D^{\mu_2} \dots D^{\mu_N} \frac{\lambda_r}{2} q(x)] - \text{TraceTerms} \,.$$

$$F_i(x, Q^2) = \sum_j \underbrace{C_i^j\left(x, \frac{Q^2}{\mu^2}\right)}_{\text{Wilson coefficients}} \otimes \underbrace{f_j(x, \mu^2)}_{\text{parton densities}}$$

Wilson coefficients: perturbative

parton densities: non-perturbative

with $$[f \otimes g](z) = \int_0^1 dz_1 \int_0^1 dz_2 \;\; \delta(z - z_1 z_2) \; f(z_1) g(z_2) \; ,$$

In Mellin-space:
Mellin-transformed function:

$$\mathbf{M}[f](N) := \int_0^1 dz \; z^{N-1} f(z) \; ,$$

Convolution becomes product:

$$\mathbf{M}[f \otimes g](N) = \mathbf{M}[f](N)\mathbf{M}[g](N)$$

Singlet quark gluon vector:

$$\mathbf{q} = \begin{pmatrix} \Sigma \\ g \end{pmatrix}, \qquad \Sigma = \sum_{r=1}^{n_f} [q_r + \overline{q}_r]$$

DGLAP evolution equation:

$$\frac{\partial \mathbf{q}(\mathbf{x}, \mathbf{Q^2})}{\partial \ln Q^2} = \mathbf{P}(x, a_s) \otimes \mathbf{q}(x, Q^2)$$

$$\mathbf{P}(x, a_s) = \begin{pmatrix} P_{qq}(x, a_s) & P_{qg}(x, a_s) \\ P_{gq}(x, a_s) & P_{gg}(x, a_s) \end{pmatrix} = \sum_{\ell=0}^{\infty} a_s^{\ell+1} \mathbf{P}_\ell(x)$$

$$\frac{d\ a_s}{d\ \ln \mu^2} = \beta(a_s) = -\sum_{\ell=0}^{\infty} a_s^{\ell+2} \beta_\ell$$

Solution of the singlet evolution equation: J. Blümlein, A. Vogt; Phys. Rev. D **58** (1998)

$$\frac{\partial \mathbf{q}(a_s, N)}{\partial a_s} = -\frac{1}{a_s}[\mathbf{R}_0(N) + \sum_{k=1}^{\infty} a_s^k \mathbf{R}_k(N)]\mathbf{q}(a_s, N)$$

with

$$\mathbf{R}_0 \equiv \frac{1}{\beta_0}\mathbf{P}_0, \qquad \mathbf{R}_k \equiv \frac{1}{\beta_0}\mathbf{P}_k - \sum_{i=1}^{k} \frac{\beta_i}{\beta_0}\mathbf{R}_{k-i}$$

Ansatz for series around lowest order: Eigenvector decomposition

$$\mathbf{q}^{LO}(a_s, N) = \left(\frac{a_s}{a_0}\right)^{-\mathbf{R}_0(N)} \mathbf{q}(a_0, N) \equiv \mathbf{L}(a_s, a_0, N)\mathbf{q}(a_0, N) :$$

$$\mathbf{q}(a_s, N) = \mathbf{U}(a_s, N)\mathbf{L}(a_s, a_0, N)\mathbf{U}^{-1}(a_0, N)\mathbf{q}(a_0, N) , \quad \mathbf{U}(a_s, N) = [1 + \sum_{k=1}^{\infty} a_s^k \mathbf{U}_k(N)]$$

$$\mathbf{R}_0 = r_- \mathbf{e}_- + r_+ \mathbf{e}_+ ,$$

$$\mathbf{e}_\pm = \frac{1}{r_\pm - r_\mp}[\mathbf{R}_0 - r_\mp \mathbf{I}] ,$$

$$r_\pm = \frac{1}{2\beta_0}\left[P_{qq}^{(0)} + P_{gg}^{(0)} \pm \sqrt{(P_{qq}^{(0)} - P_{gg}^{(0)})^2 + 4P_{qg}^{(0)} P_{gq}^{(0)}}\right]$$

LO evolution operator:

$$\mathbf{L}(a_s, a_0, N) = \mathbf{e}_-(N)\left(\frac{a_s}{a_0}\right)^{-r_-(N)} + \mathbf{e}_+(N)\left(\frac{a_s}{a_0}\right)^{-r_+(N)}$$

$$\mathbf{U}_k = -\frac{1}{k}[\mathbf{e}_-\tilde{\mathbf{R}}_k\mathbf{e}_- + \mathbf{e}_+\tilde{\mathbf{R}}_k\mathbf{e}_+] + \frac{\mathbf{e}_+\tilde{\mathbf{R}}_k\mathbf{e}_-}{r_- - r_+ - k} + \frac{\mathbf{e}_-\tilde{\mathbf{R}}_k\mathbf{e}_+}{r_+ - r_- - k}$$

z-space:

$$F_2^{Q\overline{Q}}(x,Q^2) = x\int_x^{z_{max}} \frac{dz}{z} e_Q^2 \left[H_{2,g}^S\left(z,\frac{Q^2}{m^2},\frac{m^2}{\mu^2}\right) G\left(\frac{x}{z},\mu^2\right) + H_{2,q}^{PS}\left(z,\frac{Q^2}{m^2},\frac{m^2}{\mu^2}\right)\Sigma\left(\frac{x}{z},\mu^2\right)\right]$$
$$+ x\int_x^{z_{max}} \frac{dz}{z} H_{2,q}^{NS}\left(z,\frac{Q^2}{m^2},\frac{m^2}{\mu^2}\right)\Delta\left(\frac{x}{z},\mu^2\right)$$

with flavour singlet combination of quark densities:

$$\Sigma(z,\mu^2) = \sum_{i=1}^{n_f}\left(f_i(z,\mu^2) + \bar{f}_i(z,\mu^2)\right)$$

and flavour non-singlet combination of quark densities:

$$\Delta(z,\mu^2) = \sum_{i=1}^{n_f}\left(e_i^2 - \frac{1}{n_f}\sum_{k=1}^{n_f} e_k^2\right)\left(f_i(z,\mu^2) + \bar{f}_i(z,\mu^2)\right)$$

In limit $Q^2 \gg m_Q^2$, considering all logarithmic terms & the constant term:

$$\underbrace{H^{\mathrm{S,NS}}_{(2,L),i}\left(\frac{Q^2}{m^2},\frac{m^2}{\mu^2},x\right)}_{\text{Heavy flavor Wilson coefficients}} = \underbrace{C^{S,NS}_{(2,L),k}\left(\frac{Q^2}{\mu^2},x\right)}_{\text{light-flavor Wilson coeff.}} \otimes \underbrace{A^{\mathrm{S,NS}}_{k,i}\left(\frac{m^2}{\mu^2},x\right)}_{\text{massive operator matrix elements}}$$

$$A^{\mathrm{S,NS}}_{k,i}\left(\frac{m^2}{\mu^2}\right) = \langle i|O^{\mathrm{S,NS}}_k|i\rangle = \delta_{k,i} + \sum_{l=1}^{\infty} a_s^{\ell} A^{\mathrm{S,NS},(\ell)}_{k,i}\left(\frac{m^2}{\mu^2}\right), \qquad i = q, g$$

$$C^{S,NS}_{(2,L),k}\left(\frac{Q^2}{\mu^2}\right) = \sum_{\ell=\ell_0}^{\infty} a_s^{\ell} C^{(\ell),S,NS}_{(2,L),k}\left(\frac{Q^2}{\mu^2}\right), \qquad k = g\ ,q\ ,$$

$$\widehat{C}_{(2,L),k}\left(\frac{Q^2}{\mu^2}\right) = C_{(2,L),k}\left(\frac{Q^2}{\mu^2}, N_L + N_H\right) - C_{(2,L),k}\left(\frac{Q^2}{\mu^2}, N_L\right) \ .$$

Expansion up to $O(\alpha_s^2)$: heavy flavour coefficients:

$$\begin{aligned}
H_{2,g}^{S}\left(\frac{Q^2}{m^2},\frac{m^2}{\mu^2}\right) &= a_s\left[A_{Qg}^{(1)}\left(\frac{m^2}{\mu^2}\right)+\widehat{C}_{2,g}^{(1)}\left(\frac{Q^2}{\mu^2}\right)\right] \\
&+ a_s^2\left[A_{Qg}^{(2)}\left(\frac{m^2}{\mu^2}\right)+A_{Qg}^{(1)}\left(\frac{m^2}{\mu^2}\right)\otimes C_{2,q}^{(1)}\left(\frac{Q^2}{\mu^2}\right)+\widehat{C}_{2,g}^{(2)}\left(\frac{Q^2}{\mu^2}\right)\right], \\
H_{2,q}^{PS}\left(\frac{Q^2}{m^2},\frac{m^2}{\mu^2}\right) &= a_s^2\left[A_{Qq}^{\mathrm{PS},(2)}\left(\frac{m^2}{\mu^2}\right)+\widehat{C}_{2,q}^{\mathrm{PS},(2)}\left(\frac{Q^2}{\mu^2}\right)\right], \\
H_{2,q}^{\mathrm{NS}}\left(\frac{Q^2}{m^2},\frac{m^2}{\mu^2}\right) &= a_s^2\left[A_{qq,Q}^{\mathrm{NS},(2)}\left(\frac{m^2}{\mu^2}\right)+\widehat{C}_{2,q}^{\mathrm{NS},(2)}\left(\frac{Q^2}{\mu^2}\right)\right].
\end{aligned}$$

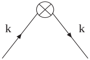
Expansion up to O(α_s^2): massive operator matrix elements:

$$
\begin{aligned}
A_{Qg}^{(1)} &= -\frac{1}{2}\widehat{P}_{qg}^{(0)} \ln\left(\frac{m^2}{\mu^2}\right) \\
A_{Qg}^{(2)} &= \frac{1}{8}\left\{\widehat{P}_{qg}^{(0)} \otimes \left[P_{qq}^{(0)} - P_{gg}^{(0)} + 2\beta_0\right]\right\} \ln^2\left(\frac{m^2}{\mu^2}\right) - \frac{1}{2}\widehat{P}_{qg}^{(1)} \ln\left(\frac{m^2}{\mu^2}\right) \\
&\quad + \overline{a}_{Qg}^{(1)}\left[P_{qq}^{(0)} - P_{gg}^{(0)} + 2\beta_0\right] + a_{Qg}^{(2)} \\
A_{Qq}^{\mathrm{PS},(2)} &= -\frac{1}{8}\widehat{P}_{qg}^{(0)} \otimes P_{gq}^{(0)} \ln^2\left(\frac{m^2}{\mu^2}\right) - \frac{1}{2}\widehat{P}_{qq}^{\mathrm{PS},(1)} \ln\left(\frac{m^2}{\mu^2}\right) + a_{Qq}^{\mathrm{PS},(2)} + \overline{a}_{Qg}^{(1)} \otimes P_{gq}^{(0)} \\
A_{qq,Q}^{\mathrm{NS},(2)} &= -\frac{\beta_{0,Q}}{4} P_{qq}^{(0)} \ln^2\left(\frac{m^2}{\mu^2}\right) - \frac{1}{2}\widehat{P}_{qq}^{\mathrm{NS},(1)} \ln\left(\frac{m^2}{\mu^2}\right) + a_{qq,Q}^{\mathrm{NS},(2)} + \frac{1}{4}\beta_{0,Q}\zeta_2 P_{qq}^{0} \ .
\end{aligned}
$$

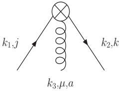
Expansion up to $O(\alpha_s^2)$: massive operator matrix elements:

$$
\begin{aligned}
A_{Qg}^{(1)} &= -\frac{1}{2}\widehat{P}_{qg}^{(0)} \ln\left(\frac{m^2}{\mu^2}\right) \\
A_{Qg}^{(2)} &= \frac{1}{8}\left\{\widehat{P}_{qg}^{(0)} \otimes \left[P_{qq}^{(0)} - P_{gg}^{(0)} + 2\beta_0\right]\right\} \ln^2\left(\frac{m^2}{\mu^2}\right) - \frac{1}{2}\widehat{P}_{qg}^{(1)} \ln\left(\frac{m^2}{\mu^2}\right) \\
&\quad + \overline{a}_{Qg}^{(1)}\left[P_{qq}^{(0)} - P_{gg}^{(0)} + 2\beta_0\right] + a_{Qg}^{(2)} \\
A_{Qq}^{\mathrm{PS},(2)} &= -\frac{1}{8}\widehat{P}_{qg}^{(0)} \otimes P_{gq}^{(0)} \ln^2\left(\frac{m^2}{\mu^2}\right) - \frac{1}{2}\widehat{P}_{qq}^{\mathrm{PS},(1)} \ln\left(\frac{m^2}{\mu^2}\right) + a_{Qq}^{\mathrm{PS},(2)} + \overline{a}_{Qg}^{(1)} \otimes P_{gq}^{(0)} \\
A_{qq,Q}^{\mathrm{NS},(2)} &= -\frac{\beta_{0,Q}}{4} P_{qq}^{(0)} \ln^2\left(\frac{m^2}{\mu^2}\right) - \frac{1}{2}\widehat{P}_{qq}^{\mathrm{NS},(1)} \ln\left(\frac{m^2}{\mu^2}\right) + a_{qq,Q}^{\mathrm{NS},(2)} + \frac{1}{4}\beta_{0,Q}\zeta_2 P_{qq}^{0} .
\end{aligned}
$$

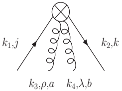
Operator insertions in light-cone expansion:



$$\equiv \quad (\Delta k)^{N}$$



$$\equiv \quad g t^a_{ij} \Delta^\mu \sum_{j=0}^{N-1} (\Delta k_1)^j (\Delta k_2)^{N-1-j}$$

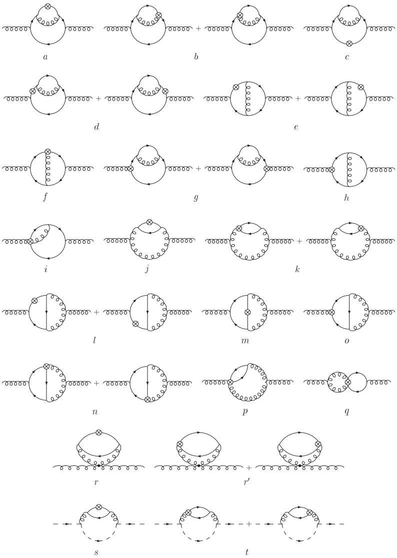


$$\equiv \quad g^2 \sum_{1\le j<l\le(N-1)} (\Delta k_2)^{j-1} (\Delta k_1 + \Delta k_4)^{l-j-1} (\Delta k_1)^{(N-1-l)} \Delta_\lambda \Delta_\rho (t^a t^b)_{kj}$$

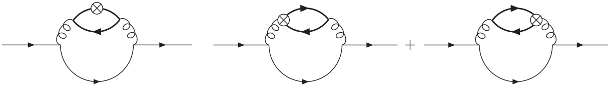
$$+\, g^2 \sum_{1\le j<l\le(N-1)} (\Delta k_2)^{j-1} (\Delta k_1 + \Delta k_3)^{l-j-1} (\Delta k_1)^{(N-1-l)} \Delta_\lambda \Delta_\rho (t^b t^a)_{kj}$$

The Calculation

Diagrams contributing to the gluonic OME $\hat{A}_{Qg}^{(2)}$:

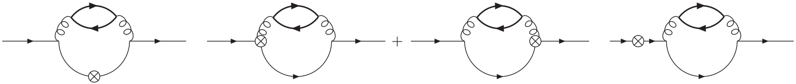


Pure singlet:



a b

Non singlet:



a b c

Our calculation:

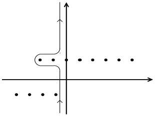
- Evaluation in Mellin space $\rightarrow$ Analytic results for general value of Mellin N $\rightarrow$ harmonic sums:

$$S_{a_1,\ldots,a_m}(N) = \sum_{n_1=1}^{N} \sum_{n_2=1}^{n_1} \ldots \sum_{n_m=1}^{n_{m-1}} \frac{(\mathrm{sign}(a_1))^{n_1}}{n_1^{|a_1|}} \frac{(\mathrm{sign}(a_2))^{n_2}}{n_2^{|a_2|}} \ldots \frac{(\mathrm{sign}(a_m))^{n_m}}{n_m^{|a_m|}}$$

$$N \in \mathbb{N}, \ \forall\, \ell, \ a_\ell \in \mathbb{Z} \setminus 0$$

- use of Mellin-Barnes integrals

$$\frac{1}{(A+B)^{-\nu}} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} d\sigma A^{\sigma} B^{-\nu-\sigma} \frac{\Gamma(-\sigma)\Gamma(\nu+\sigma)}{\Gamma(\nu)}$$

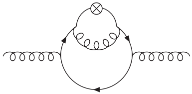


$\rightsquigarrow$ numeric check & some analytic results

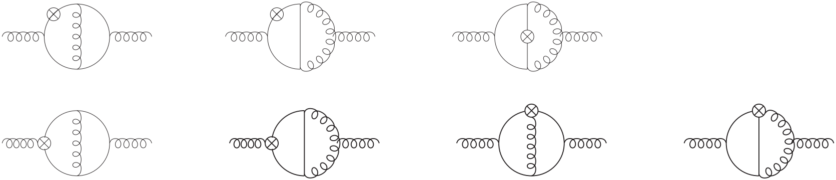
- use of hypergeometric functions for general analytic results

$${}_PF_Q\left[\begin{array}{c} (a_1)\ldots(a_P) \\ (b_1)\ldots(b_Q) \end{array}; z\right] = \sum_{i=0}^{\infty} \frac{(a_1)_i \ldots (a_F)_i}{(b_1)_i \ldots (b_Q)_i} \frac{z^i}{\Gamma(i+1)}, \quad (c)_n = \frac{\Gamma(c+n)}{\Gamma(c)}.$$

- {Diagrams: a-d, g, j-k, p-t} + PS + NS: 2-loop diagrams with 2-point 1-loop insertions



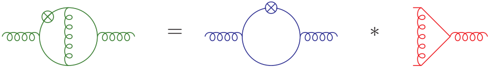
- {Diagrams: e,f,h,l,m,n,o}: genuine 2-loop diagrams
  → cancellation of propagators (→ also reduced diagrams):



Genuine two-loop diagrams via Mellin-Barnes integrals: Example, Diagram e:

I.B., S. Weinzierl, Eur. Phys. J. C 32 (2003) (massless case);

I.B., J. Blümlein and S. Klein, Nucl. Phys. Proc. Suppl. **160**, 85 (2006);



$$I_{e,\nu_1} = \frac{(\Delta p)^{N-1}}{(4\pi)^D (m^2)^{\nu_{12345}-D}} \frac{1}{(2\pi i)^2} \frac{(-1)^{\nu_{12345}+1}}{\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_5)\Gamma(D-\nu_{235})}$$

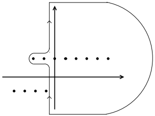
$$\times \int_{\gamma_1-i\infty}^{\gamma_1+i\infty} d\sigma \int_{\gamma_2-i\infty}^{\gamma_2+i\infty} d\tau \, \Gamma(-\sigma)\Gamma(\nu_3+\sigma) \frac{\Gamma(-\sigma+\nu_4+N-1)}{\Gamma(-\sigma+\nu_4)} \Gamma(-\tau)\Gamma(\nu_2+\tau)$$

$$\times \frac{\Gamma(\sigma+\tau+\nu_{235}-D/2)\Gamma(\sigma+\tau+\nu_5)}{\Gamma(\sigma+\tau+\nu_{23})\Gamma(-\sigma-\tau+D-\nu_{23}-2\nu_5)} \frac{\Gamma(-\sigma-\tau+\nu_{14}-D/2)}{\Gamma(-\sigma-\tau+\nu_{14}+N-1)}$$

N	2	3	4	5
$\mathrm{I}_{e,1}$	+0.49999	+0.31018	+0.21527	+0.16007
$\mathrm{I}_{e,2}$	−0.09028	−0.04398	−0.02519	−0.01596

M. Czakon, [MB], Comput. Phys. Commun. **175** (2006)

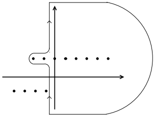
Residue Theorem and Barnes Lemma:



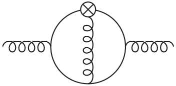
$$I_{e,1} \Rightarrow \frac{\Gamma(N+1)}{\Gamma(1+\varepsilon)} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+2+N)}$$
$$\times \left[\Gamma(-\varepsilon/2)\Gamma(1+\varepsilon/2) \frac{\Gamma(j+1+\varepsilon)\Gamma(j+1-\varepsilon/2)}{\Gamma(j+1+\varepsilon/2)\Gamma(j+2+N)} \frac{\Gamma(k+j+1+N)}{\Gamma(k+j+2)}\right.$$
$$\left. + \Gamma(\varepsilon/2)\Gamma(1-\varepsilon/2) \frac{\Gamma(j+1-\varepsilon)\Gamma(j+1+\varepsilon/2)}{\Gamma(j+1)\Gamma(j+2-\varepsilon/2+N)} \frac{\Gamma(k+j+1-\varepsilon/2+N)}{\Gamma(k+j+2-\varepsilon/2)}\right].$$

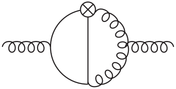
$\mathrm{I}_{e,1}$	$\frac{S_1^2(N)+3S_2(N)}{2N(N+1)}$
$\mathrm{I}_{e,2}$	$\frac{S_1(N)-S_2(N)-S_{1,1}(N)}{N(N+1)(N+2)} - \frac{1}{(N+1)^2(N+2)}$

Residue Theorem and Barnes Lemma:



$$
\begin{aligned}
I_{e,1} \Rightarrow \quad & \frac{\Gamma(N+1)}{\Gamma(1+\varepsilon)} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+2+N)} \\
\times \quad & \Big[\Gamma(-\varepsilon/2)\Gamma(1+\varepsilon/2) \frac{\Gamma(j+1+\varepsilon)\Gamma(j+1-\varepsilon/2)}{\Gamma(j+1+\varepsilon/2)\Gamma(j+2+N)} \frac{\Gamma(k+j+1+N)}{\Gamma(k+j+2)} \\
+ \quad & \Gamma(\varepsilon/2)\Gamma(1-\varepsilon/2) \frac{\Gamma(j+1-\varepsilon)\Gamma(j+1+\varepsilon/2)}{\Gamma(j+1)\Gamma(j+2-\varepsilon/2+N)} \frac{\Gamma(k+j+1-\varepsilon/2+N)}{\Gamma(k+j+2-\varepsilon/2)}\Big] \,.
\end{aligned}
$$

f 

n 

Way 2: Hypergeometric functions: Example, Diagram e again:

$$I_{e,a} := \iint \frac{dq\ dk}{(2\pi)^{2D}} \frac{(\Delta q)^{N-1}}{(q^2-m^2)^a((q-p)^2-m^2)(k^2-m^2)((k-p)^2-m^2)((k-q)^2)}$$

$$= \frac{(-1)^{N+a}\Gamma(4+a-D/2)}{(4\pi)^{D/2}} \int_0^1 dydz \int_0^1 dudvdw\ \delta(1-u-v-w)$$

$$\int \frac{dq}{(2\pi)^D} \frac{(\Delta q)^{N-1} w^{2-D/2} (z(1-z))^{D/2-3}(1-z)u^{a-1}}{(q^2+m^2(u+v+\frac{w}{z})-2qp(v+wy))^{4+a-D/2}}$$

$$I_{e,1} = \frac{S_\varepsilon^2}{(4\pi)^4(m^2)^{1-\varepsilon}} \frac{(\Delta p)^{N-1}}{N(N+1)} \exp\left\{\sum_{i=2}^{\infty} \zeta_i \frac{\varepsilon^i}{i}\right\} \Bigg\{$$

$$B(\varepsilon/2+1, 1-\varepsilon/2)B(1,-\varepsilon/2)\ {}_3F_2\left[\begin{matrix} 1-\varepsilon, 1, 1+\varepsilon/2 \\ 2, 1-\varepsilon/2 \end{matrix}\ ;1\right]$$

$$-\ B(\varepsilon/2+1, 1-\varepsilon/2)B(1, N+1-\varepsilon/2)\ {}_3F_2\left[\begin{matrix} 1-\varepsilon, 1, 1+\varepsilon/2 \\ 2, N+2-\varepsilon/2 \end{matrix}\ ;1\right]$$

$$-\ B(\varepsilon/2+1, 1-\varepsilon/2)B(N+2,-\varepsilon/2)\ {}_3F_2\left[\begin{matrix} 1-\varepsilon, N+2, 1+\varepsilon/2 \\ 2, N+2-\varepsilon/2 \end{matrix}\ ;1\right]\Bigg\}$$

Way 2: Hypergeometric functions: Example, Diagram e again:

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$$I_{e,1} = \frac{S_\varepsilon^2}{(4\pi)^4(m^2)^{1-\varepsilon}}(\Delta p)^{N-1}\left\{\frac{S_1^2(N)+3S_2(N)}{2N(N+1)}\right\} + O(\varepsilon)$$

Results

$$A_e^{Qg} = T_R\left[C_F - \frac{C_A}{2}\right]\Bigg\{\frac{1}{\varepsilon^2}\frac{16(N+3)}{(N+1)^2} + \frac{1}{\varepsilon}\Bigg[-\frac{8(N+2)}{N(N+1)}S_1(N)$$

$$-8\frac{3N^3+9N^2+12N+4}{N(N+1)^3(N+2)}\Bigg] - 2\frac{9N^4+40N^3+71N^2-12N-36}{N(N+1)^2(N+2)(N+3)}S_2(N)$$

$$-2\frac{N^3-N^2-8N-36}{N(N+1)(N+2)(N+3)}S_1^2(N) + 4\frac{(N+3)}{(N+1)^2}\zeta_2$$

$$+4\frac{4N^5+19N^4+31N^3-30N^2-44N-24}{N^2(N+1)^2(N+2)(N+3)}S_1(N)$$

$$+\frac{4P_4(N)}{N^2(N+1)^4(N+2)^2(N+3)}\Bigg\},$$

$$P_4(N) = 16N^7+111N^6+342N^5+561N^4+536N^3+354N^2+152N+24\,.$$

$$a_{Qg}^{(2)}(N) = 4C_FT_R\Bigg\{\frac{N^2+N+2}{N(N+1)(N+2)}\Bigg[-\frac{1}{3}S_1^3(N-1)+\frac{4}{3}S_3(N-1)-S_1(N-1)S_2(N-1)-2\zeta_2S_1(N-1)\Bigg]$$
$$+\frac{2}{N(N+1)}S_1^2(N-1)+\frac{N^4+16N^3+15N^2-8N-4}{N^2(N+1)^2(N+2)}S_2(N-1)+\frac{3N^4+2N^3+3N^2-4N-4}{2N^2(N+1)^2(N+2)}\zeta_2$$
$$+\frac{N^4-N^3-16N^2+2N+4}{N^2(N+1)^2(N+2)}S_1(N-1)+\frac{P_2(N)}{2N^4(N+1)^4(N+2)}\Bigg\}$$
$$+4C_AT_R\Bigg\{\frac{N^2+N+2}{N(N+1)(N+2)}\Bigg[4\mathbf{M}\left[\frac{\mathrm{Li}_2(x)}{1+x}\right](N+1)+\frac{1}{3}S_1^3(N)+3S_2(N)S_1(N)$$
$$+\frac{8}{3}S_3(N)+\beta''(N+1)-4\beta'(N+1)S_1(N)-4\beta(N+1)\zeta_2+\zeta_3\Bigg]-\frac{N^3+8N^2+11N+2}{N(N+1)^2(N+2)^2}S_1^2(N)$$
$$-2\frac{N^4-2N^3+5N^2+2N+2}{(N-1)N^2(N+1)^2(N+2)}\zeta_2-\frac{7N^5+21N^4+13N^3+21N^2+18N+16}{(N-1)N^2(N+1)^2(N+2)^2}S_2(N)$$
$$-\frac{N^6+8N^5+23N^4+54N^3+94N^2+72N+8}{N(N+1)^3(N+2)^3}S_1(N)-4\frac{(N^2-N-4)}{(N+1)^2(N+2)^2}\beta'(N+1)$$
$$+\frac{P_3(N)}{(N-1)N^4(N+1)^4(N+2)^4}\Bigg\}.$$

$$\mathbf{M}\left[\frac{\mathrm{Li}_2(x)}{1+x}\right](N+1)=\zeta_2\beta(N+1)+(-1)^{N+1}\left[S_{-2,1}(N)+\frac{5}{8}\zeta_3\right], \qquad \beta(x)=\frac{1}{2}\left[\psi\left(\frac{x+1}{2}\right)-\psi\left(\frac{x}{2}\right)\right]$$

$$a_{Qq}^{\mathrm{PS},(2)}(N) = T_R C_F \left\{ -4\frac{(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)}\left(2S_2(N)+\zeta_2\right) + \frac{4P_4(N)}{(N-1)N^4(N+1)^4(N+2)^3} \right\},$$

$$a_{qq,Q}^{\mathrm{NS},(2)}(N) = C_F T_R \left\{ -\frac{8}{3}S_3(N) - \frac{8}{3}\zeta_2 S_1(N) + \frac{40}{9}S_2(N) + 2\frac{3N^2+3N+2}{3N(N+1)}\zeta_2 - \frac{224}{27}S_1(N) + \frac{219N^6+657N^5+1193N^4+763N^3-40N^2-48N+72}{54N^3(N+1)^3} \right\}$$

$$\begin{aligned}
P_1(N) &= N^9+6N^8+15N^7+25N^6+36N^5+85N^4+128N^3+104N^2+64N+16\,,\\
P_2(N) &= 12N^8+54N^7+136N^6+218N^5+221N^4+110N^3-3N^2-24N-4\,,\\
P_3(N) &= 2\,N^{12}+20\,N^{11}+86\,N^{10}+192\,N^9+199\,N^8-N^7-297\,N^6-495\,N^5\\
&\quad -514\,N^4-488\,N^3-416\,N^2-176\,N-32\,.
\end{aligned}$$

Comparison

First Calculation to $O(\alpha_S^2)$:
M. Buza et al., Nucl. Phys. B 472 (1996) 611 → used standard methods

- ⇝ Integration-by-parts method
- ⇝ direct integration of individual Feynman-parameter integrals in z-space
- ⇒ combinations of Nielsen integrals

$$S_{p,n}(x) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \frac{dz}{z} \ln^{n-1}(z) \ln^p(1-zx)$$

with partly complicated arguments, e.g.:

$$\int_0^1 dy \frac{1-x}{1-y(1-x)} \ln(1-y) \ln\left(1+y\frac{1-x}{x}\right)$$
$$= \frac{1}{2} \ln^3(x) + 2\,S_{1,2}(1-x) - 3\,\mathrm{Li}_3(-x) - \zeta_2 \ln(x) + \ln(x)\,\mathrm{Li}_2(-x) - \frac{5}{2}\zeta_3$$
$$+ 2 \ln(1+x)\,\mathrm{Li}_2(-x) + \zeta_2 \ln(1+x) + \ln(x) \ln^2(1+x) - \frac{1}{2} \ln^2(x) \ln(1+x)$$
$$+ 2\,S_{1,2}(-x) - 2\,\mathrm{Li}_3(1-x) + 2 \ln(x)\,\mathrm{Li}_2(1-x) - 2\,\mathrm{Li}_3\left(-\frac{1-x}{1+x}\right) + 2\,\mathrm{Li}_3\left(\frac{1-x}{1+x}\right)$$

Functions contributing to the results in z–space:

$\delta(1-x)$	1	$\ln(x)$	$\ln^2(x)$	$\ln^3(x)$
$\ln(1-x)$	$\ln^2(1-x)$	$\ln^3(1-x)$	$\ln(x)\ln(1-x)$	$\ln(x)\ln^2(1-x)$
$\ln^2(x)\ln(1-x)$	$\ln(1+x)$	$\ln(x)\ln(1+x)$	$\ln^2(x)\ln(1+x)$	$\mathrm{Li}_2(1-x)$
$\ln(x)\mathrm{Li}_2(1-x)$	$\ln(1-x)\mathrm{Li}_2(1-x)$	$\mathrm{Li}_3(1-x)$	$S_{1,2}(1-x)$	$S_{1,2}(-x)$
$\frac{1}{1-x}$	$\frac{1}{1+x}$	$\frac{\ln(x)}{1-x}$	$\frac{\ln^2(x)}{1-x}$	$\frac{\ln^3(x)}{1-x}$
$\frac{\ln(x)}{1+x}$	$\frac{\ln^2(x)}{1+x}$	$\frac{\ln^3(x)}{1+x}$	$\frac{\ln(1+x)}{1+x}$	$\frac{\ln(x)\ln(1+x)}{1+x}$
$\frac{\ln(x)\ln^2(1+x)}{1+x}$	$\frac{\ln^2(x)\ln(1+x)}{1+x}$	$\frac{\ln(x)\ln(1-x)}{1-x}$	$\frac{\ln(x)\ln^2(1-x)}{1-x}$	$\frac{\ln(1-x)\mathrm{Li}_2(x)}{1-x}$
$\frac{\mathrm{Li}_2(1-x)}{1-x}$	$\frac{\ln(x)\mathrm{Li}_2(1-x)}{1-x}$	$\frac{\ln(x)\mathrm{Li}_2(1-x)}{1+x}$	$\frac{\ln(1+x)\mathrm{Li}_2(-x)}{1+x}$	$\ln(1+x)\mathrm{Li}_2(-x)$
$\mathrm{Li}_2(-x)$	$\frac{\mathrm{Li}_2(-x)}{1+x}$	$\frac{\ln(x)\mathrm{Li}_2(-x)}{1+x}$	$\frac{\mathrm{Li}_3(1-x)}{1-x}$	$\frac{\mathrm{Li}_3(-x)}{1+x}$
$\frac{S_{1,2}(1-x)}{1-x}$	$\frac{S_{1,2}(1-x)}{1+x}$	$\frac{S_{1,2}(-x)}{1+x}$		

Complexity of the results in Mellin space and in z–space for each diagram:

Diagram	S_1	S_2	S_3	S_{-2}	S_{-3}	$S_{2,1}$	$S_{-2,1}$	# z-space fct.
A		+						8
B	+	+	+			+		10
C		+						4
D	+	+						5
E	+	+						9
F	+	+	+			+		24
G	+	+						6
H	+	+						7
I	+	+	+	+	+	+	+	20
J		+						7
K		+						7
L	+	+	+			+		13
M		+						7
N	+	+	+	+	+	+	+	38
O	+	+	+			+		13
P	+	+	+			+		14
S		+						7
T		+						7
PS_a		+						
PS_b		+						7
NS_a								
NS_b	+	+	+					5
$\sum$	+	+	+	+	+		+	48

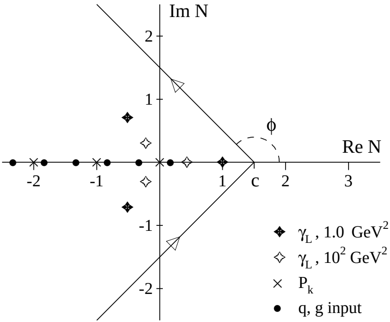
$\Rightarrow$ 6 basic functions:

5 of them single harmonic sums, which are related algebraically and by differentiation

$\Rightarrow$ 2 basic objects

Mellin-Inversion

Inversion from Mellin-space to z-space: J. Blümlein, Ancont



Continuation of harmonic sums:

$S_1(N) = \Psi(N+1) + \gamma$,
etc.

$$xF_2^{Q\bar{Q}}(x, Q^2) = \int_0^\infty dz \, \mathrm{Im} \left[e^{i\Phi} x^{-c(z)} \, F_2^{Q\bar{Q}}(c(z), Q^2) \right],$$
$$c(z) = c_0 + ze^{i\Phi}$$

Conclusions

Calculation of heavy-quark effects in QCD Wilson-coefficients in the asymptotic regime $Q^2 \gg m_{Q^2}$:

- Calculation in Mellin-space
- Use of Mellin-Barnes integrals (easy numeric check) and generalized hypergeometric functions
- The results are obtained in terms of nested harmonic sums.

$\Rightarrow$ Both, the Mellin-Barnes method and Mellin-space representations are essential to achieve the obtained simplification $\rightarrow$ algebraic & structural relations of harmonic sums.

- Calculation of the constant term of the massive Operator Matrix Elements $\rightarrow$ full agreement with results of van Neerven et al.
- Comparison to exact order α_s^2 (α_s^3) result: asymptotic formulae valid for $Q^2 \geq 20$ $(Gev/c)^2$ in case of $F_{2,c}(x, Q^2, m_c^2)$ and $Q^2 \geq 600$ $(Gev/c)^2$ for $F_{L,c}(x, Q^2, m_c^2)$