Calculation of power corrections in HQET

(m_B, f_B)

with

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SFB / TR 9

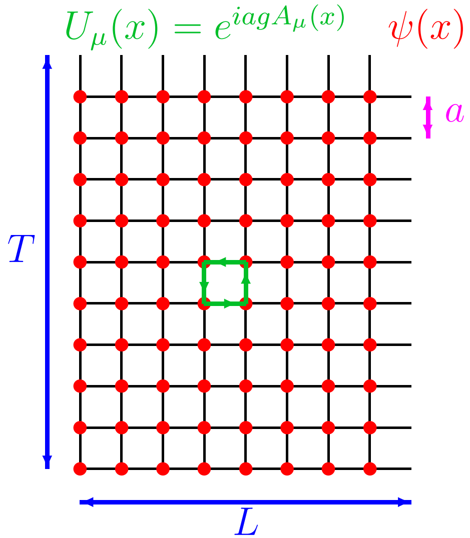
Outline

- Motivations
- Effective Theory
- Strategy
- Implementation
- Results

Motivations

Lattice QCD

- Euclidian space time $\longrightarrow$ finite discrete set of points



  - Quark fields on these points (sites)
  - Gauge fields on the links

- Provides both UV and IR cuttoffs $\Rightarrow$ well defined path integral

- # dof is finite and integrand positive $\Rightarrow$ path integral evaluated numerically by stochastic methods

$\Rightarrow$ Lattice QCD allows the computation of hadronic observables, **non perturbatively**, directly from the QCD Lagrangian (**first principles**)

Lattice QCD - In practice

Source of errors

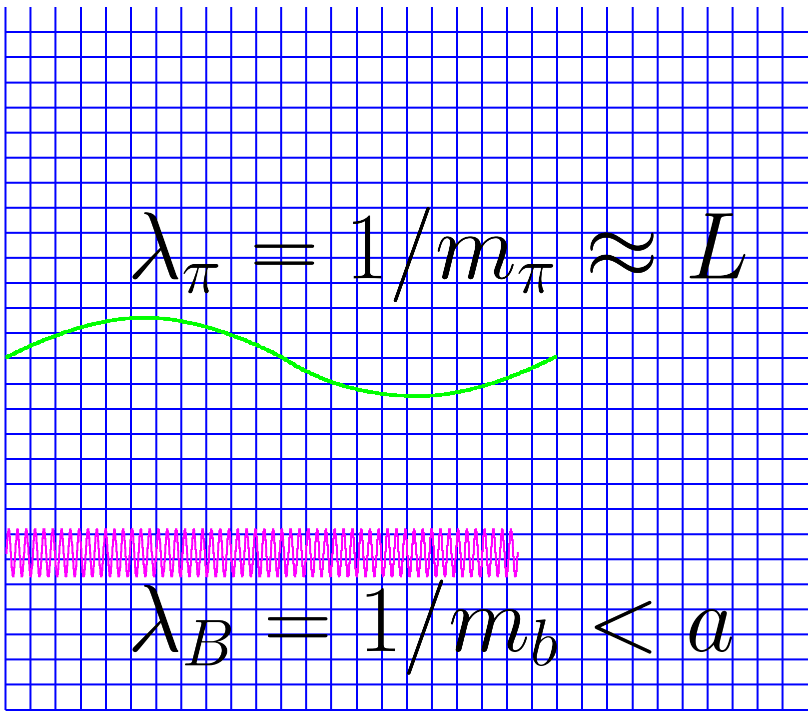
- Finite number of statistics.
  Statistical errors decrease $\propto \frac{1}{\sqrt{N_{\rm conf}}}$
- Systematic errors
  - Finite volume
    Contaminations decrease exponentially if $L \gg \frac{4}{M_\pi}$
  - Finite lattice spacing
    Discretizations errors $\propto (am_{\rm q})^\alpha$
    $\Rightarrow$ choose a and m_q such as $(am_q) \leq 1$
  - (Quenched approximation)

Nowdays calculations :

$$\begin{aligned} L &\simeq 1-3 \text{ fm} \\ a^{-1} &\simeq 2-4 \text{ GeV} \\ m_{\rm q} &\leq m_{\rm c} \end{aligned}$$

Light and heavy particules

Take a large lattice as it is possible in the quenched approximation

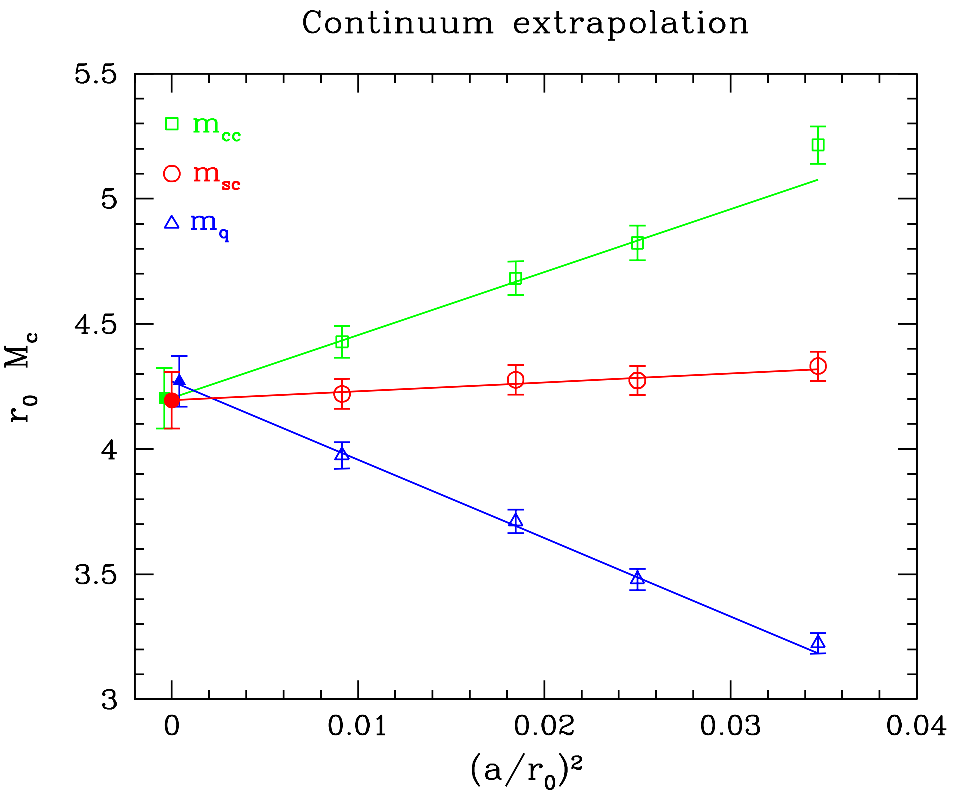


light quarks are too light
$\rightarrow$ treat by an extrapolation

b-quark is too heavy

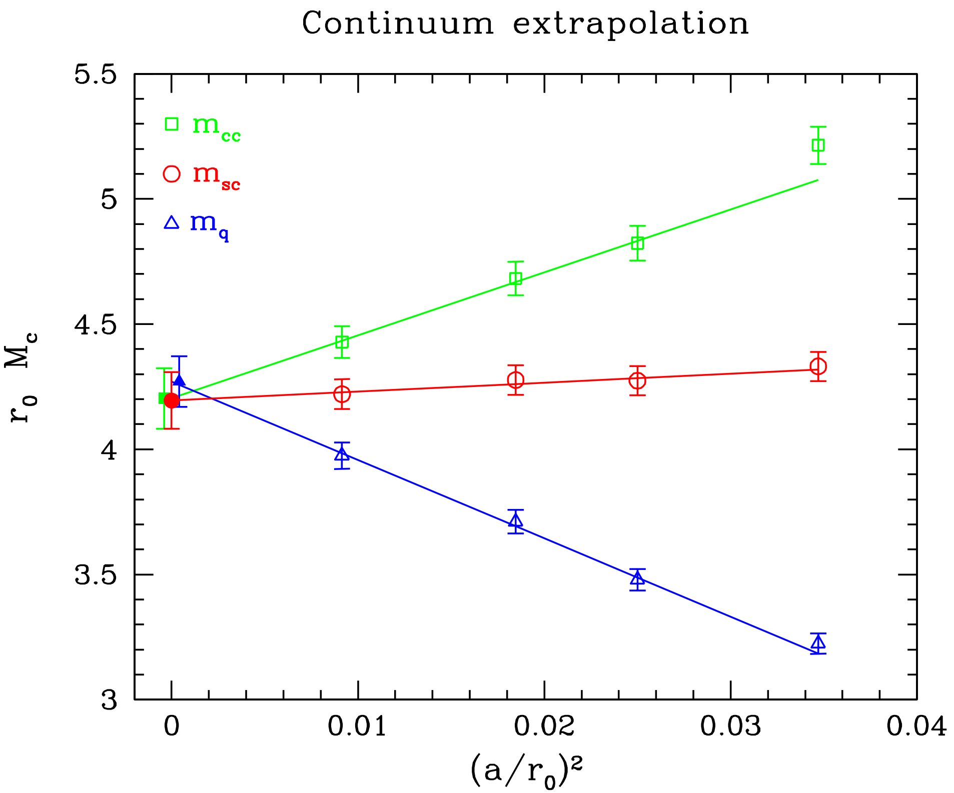
Need an effective theory for the b-quark: HQET

`E. Eichten, 1988; E. Eichten & B. Hill 1990`



The simulation of the charm mass is just doable ...

The charm quark



The simulation of the charm mass is just doable ...

... and $M_b \simeq 4M_c$.

Effective theory: HQET

NP formulation of HQET [Heitger & Sommer 03]

Action of the effective theory on a lattice [Eichten & Hill]

$$S_{\rm HQET} = a^4 \sum_x \{\overline{\psi}_{\rm h}(x)[D_0 + \delta m]\psi_{\rm h}(x) + \sum_{\nu=1}^{n} \mathcal{L}^{(\nu)}(x)\}$$

with $P_+\psi_{\rm h} = \psi_{\rm h}\,, \quad \overline{\psi}_{\rm h}P_+ = \overline{\psi}_{\rm h}\,, \quad P_+ = \frac{1}{2}(1+\gamma_0)$, and

$$\mathcal{L}^{(\nu)}(x) = \sum_i \omega_i^{(\nu)} \mathcal{L}_i^{(\nu)}(x) \qquad\qquad \omega_i^{(\nu)} \propto (1/m)^\nu$$

At the next to leading order

$$\begin{aligned} \mathcal{L}_1^{(1)} &= \overline{\psi}_{\rm h}(-\sigma\cdot\mathbf{B})\psi_{\rm h} & \qquad \mathcal{L}_2^{(1)} &= \overline{\psi}_{\rm h}(-\tfrac{1}{2}\mathbf{D}^2)\psi_{\rm h} \\ &\equiv -\mathcal{O}_{\rm spin} & &\equiv -\mathcal{O}_{\rm kin} \\ \omega_1^{(1)} &= \omega_{\rm spin} & \omega_1^{(2)} &= \omega_{\rm kin} \end{aligned}$$

Effective fields: time component of axial current in effective theory

$$A_0^{\mathrm{HQET}}(x) = \alpha_0^{(0)} A_0^{\mathrm{stat}}(x) + \sum_{\nu=1}^{n} \mathcal{A}_0^{(\nu)}(x)$$

$$\mathcal{A}^{(\nu)}(x) = \sum_i \alpha_i^{(\nu)} \mathcal{A}_i^{(\nu)}(x)\,,$$

with coefficients $\alpha_i^{(\nu)} \propto (1/m)^\nu$

To maintain $\mathcal{O}(a)$ improvement in the static approximation and to implement the $1/m$ correction

$$A_0^{\mathrm{stat}}(x) = \overline{\psi}_{\mathrm{l}}(x)\gamma_0\gamma_5\psi_{\mathrm{h}}(x)$$

$$A_0^{(1)}(x) = \overline{\psi}_{\mathrm{l}}\frac{1}{2}(\overleftarrow{D}_i + \overleftarrow{D}_i^*)\gamma_i\gamma_5\psi_{\mathrm{h}}$$

$\Rightarrow$ static theory is $\mathcal{O}(a)$-improved

Green functions (I)

Under the path integral: expand in $1/m \;\Rightarrow\; \mathcal{L}^{(\nu)}(x)$ only as insertions (renormalizability $\equiv$ continuum limit)

$$\langle \mathcal{O} \rangle \;=\; \mathcal{Z}^{-1} \int \mathrm{D}\phi \, \mathrm{e}^{-S_{\mathrm{light}} - a^4 \sum_x \overline{\psi}_{\mathrm{h}}(x)[D_0 + \delta m]\psi_{\mathrm{h}}(x)} \; \mathcal{O} \times \left\{ 1 - a^4 \sum_x \mathcal{L}^{(1)}(x) + \ldots \right\}$$

$$\equiv \; \langle [1 - a^4 \sum_x \mathcal{L}^{(1)}(x)] \; \mathcal{O} \; \rangle^{\mathrm{stat}}$$

- Coefficients $\omega_i^{(\nu)}$, $\alpha_i^{(\nu)}$ have to cancel power divergences $\Rightarrow$ have to be determined **non-perturbatively**
- then and only then: existence of the continuum limit with universality (independence of details of the regularization)
- proper asymptotic expansion in $1/m$

Green functions (II)

At the next to leading order, the Lagrangian of HQET is

$$\begin{aligned} \mathcal{L}_{\rm HQET} &= \mathcal{L}_{\rm stat}(x) - \omega_{\rm spin}\mathcal{O}_{\rm spin}(x) - \omega_{\rm kin}\mathcal{O}_{\rm kin}(x) \\ &\mathcal{O}_{\rm spin} = \bar{\psi}_{\rm h}\sigma\mathbf{B}\psi_{\rm h}\,, \quad \mathcal{O}_{\rm kin} = \bar{\psi}_{\rm h}\mathbf{D}^2\psi_{\rm h}\,, \end{aligned}$$

and the correlation functions

$$\begin{aligned} \langle\mathcal{O}\rangle &= \langle\mathcal{O}\rangle^{\rm stat} + \omega_{\rm kin}\sum_x \langle\mathcal{O}\mathcal{O}_{\rm kin}(x)\rangle^{\rm stat} + \omega_{\rm spin}\sum_x \langle\mathcal{O}\mathcal{O}_{\rm spin}(x)\rangle^{\rm stat} \\ &= \langle\mathcal{O}\rangle^{\rm stat} + \omega_{\rm kin}\,\langle\mathcal{O}\rangle^{\rm kin} + \omega_{\rm spin}\,\langle\mathcal{O}\rangle^{\rm spin} \end{aligned}$$

Strategy

Non perturbative matching of HQET and QCD

The coefficients $m_{\mathrm{bare}}, \omega_{\mathrm{kin}}, \omega_{\mathrm{spin}}, \ldots$ of HQET needed to be fixed non perturbatively.
This is achieved by the matching with QCD

$$\Phi_i^{\mathrm{QCD}} = \Phi_i^{\mathrm{HQET}} \qquad i = 1, \ldots, N_{\mathrm{HQET}}$$

This non perturbative matching requires to be able to simulate the heavy quark with finite mass.

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$$\Phi_i^{\mathrm{QCD}}(L_1) = \Phi_i^{\mathrm{HQET}}(L_1) \qquad i = 1, \ldots, N_{\mathrm{HQET}}$$

This non perturbative matching requires to be able to simulate the heavy quark with finite mass.
$\Rightarrow$ In a finite volume ($L_1 \simeq 0.4\mathrm{fm}$), with $am_{\mathrm{b}} \ll 1$.

- Choose $\Phi(L) = L\Gamma$, where $\Gamma =$“ finite volume meson mass “

b quark mass, the static case

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experiment

Lattice with $am_{\rm q} \ll 1$

$m_{\rm B} = 5.4\,{\rm GeV}$ ↓ $\Gamma^{\rm stat}(L_2)$

$\Gamma(L_1, M)$ ↓ $\Gamma^{\rm stat}(L_1)$

$L_2 = 2L_1$

$u_i = \bar{g}^2(L_i)$

$\Gamma^{\rm stat}(L_2) \xleftarrow{\sigma_{\rm m}(u_1)} \Gamma^{\rm stat}(L_1)$

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- Choose $\Phi(L) = L\Gamma$, where $\Gamma =$" finite volume meson mass "

  Matching $\Gamma^{\mathrm{QCD}}(L_1, M) = \Gamma^{\mathrm{stat}}(L_1) + m_{\mathrm{bare}}$

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$$m_{\rm B} = E^{\rm stat} - \Gamma^{\rm stat}(L_2) + \underbrace{\Gamma^{\rm stat}(L_2) + m_{\rm bare}}_{\Gamma^{\rm QCD}(L_2, M)}$$

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- Define $\sigma^{\rm stat}(u_i) = 2L[\Gamma^{\rm stat}(2L) - \Gamma^{\rm stat}(L)]|_{u_i=\bar{g}^2(L_i)}$

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- Impose $m_{\rm B} = m_{\rm B}^{\rm exp}$ and solve this equation for $M_{\rm b}$, the RGI b quark mass.

***b* quark mass, the $1/m$ correction**

At the NLO, in infinite volume

$$m_{\rm B} = E^{\rm stat} + m_{\rm bare} + \omega_{\rm kin} E^{\rm kin} + \omega_{\rm spin} E^{\rm spin}$$

$\Rightarrow$ Need three observables Φ_1, Φ_2, Φ_3.

b quark mass, the $1/m$ correction

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Or, consider the spin-averaged B meson $\Rightarrow$ $\omega_{\rm spin}$ cancels

$$m_{\rm B}^{\rm av} \equiv \frac{1}{4} m_{\rm B} + \frac{3}{4} m_{\rm B}^{*} = E^{\rm stat} + m_{\rm bare} + \omega_{\rm kin} E^{\rm kin}$$

$\Rightarrow$ Need two observables Φ_1, Φ_2, and the spin splitting term becomes a separate issue.

b quark mass, the $1/m$ correction

At the NLO, in infinite volume

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Which observables ?

Implementation

Correlation functions (I)

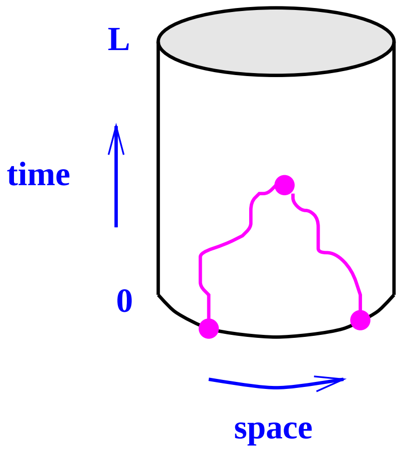
Implementation: Schrödinger functional of size $T \times L^3$

- Dirichlet boundary conditions in time (at $x_0 = 0$ and $x_0 = T$)
- Periodic boundary conditions in space, up to a phase $\Psi(x + \hat{k}L) = e^{i\theta}\Psi(x)$.

Consider boundary to current correlators in QCD

$$f_{\rm A}(x_0) = -\frac{a^6}{2}\sum_{\mathbf{y},\mathbf{z}} \left\langle (A_{\rm I})_0(x)\,(\bar{\zeta}_{\rm b}(\mathbf{y})\gamma_5\zeta_{\rm l}(\mathbf{z}))\right\rangle$$

$$k_{\rm V}(x_0) = -\frac{a^6}{6}\sum_{\mathbf{y},\mathbf{z},k} \left\langle (V_{\rm I})_k(x)\,(\bar{\zeta}_{\rm b}(\mathbf{y})\gamma_k\zeta_{\rm l}(\mathbf{z}))\right\rangle$$



Correlation functions (II)

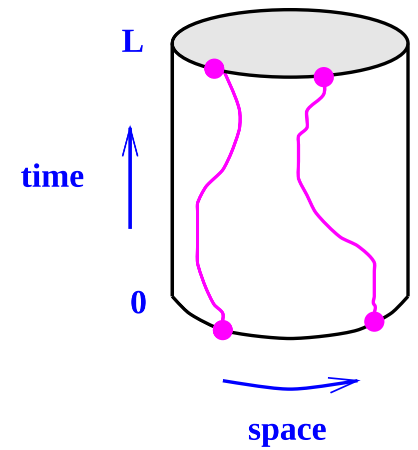
In HQET, at the next to leading order

$$[f_{\rm A}]_{\rm R} \;=\; Z_{\rm A}^{\rm HQET} Z_{\zeta_{\rm h}} Z_{\zeta} {\rm e}^{-m_{\rm bare} x_0} \left\{ f_{\rm A}^{\rm stat} + c_{\rm A}^{\rm HQET} f_{\delta{\rm A}}^{\rm stat} + \omega_{\rm kin} f_{\rm A}^{\rm kin} + \omega_{\rm spin} f_{\rm A}^{\rm spin} \right\}$$

$$[k_{\rm V}]_{\rm R} \;=\; -Z_{\rm V}^{\rm HQET} Z_{\zeta_{\rm h}} Z_{\zeta} {\rm e}^{-m_{\rm bare} x_0} \left\{ f_{\rm A}^{\rm stat} + c_{\rm V}^{\rm HQET} f_{\delta{\rm A}}^{\rm stat} + \omega_{\rm kin} f_{\rm A}^{\rm kin} - \tfrac{1}{3}\omega_{\rm spin} f_{\rm A}^{\rm spi}\right.$$

Other possibility :

Consider boundary to boundary correlators



$$f_1 \quad = \quad -\frac{a^{12}}{2L^6} \sum_{\mathbf{u},\mathbf{v},\mathbf{y},\mathbf{z}} \left\langle \left(\bar{\zeta}_{\rm l}{}'(\mathbf{u})\gamma_5\zeta'_{\rm b}(\mathbf{v})\right)\ \left(\bar{\zeta}_{\rm b}(\mathbf{y})\gamma_5\zeta_{\rm l}(\mathbf{z})\right)\right\rangle \qquad \text{in QCD}$$

$$\rightarrow \quad Z_{\zeta_{\rm h}} Z_{\zeta} {\rm e}^{-m_{\rm bare}T} \left\{ f_1^{\rm stat} + \omega_{\rm kin} f_1^{\rm kin} + \omega_{\rm spin} f_1^{\rm spin} \right\} \qquad \text{in HQET}$$

$$k_1 \quad = \quad -\frac{a^{12}}{2L^6} \sum_{\mathbf{u},\mathbf{v},\mathbf{y},\mathbf{z},k} \left\langle \left(\bar{\zeta}_{\rm l}{}'(\mathbf{u})\gamma_k\zeta'_{\rm b}(\mathbf{v})\right)\ \left(\bar{\zeta}_{\rm b}(\mathbf{y})\gamma_k\zeta_{\rm l}(\mathbf{z})\right)\right\rangle \qquad \text{in QCD}$$

$$\rightarrow \quad Z_{\zeta_{\rm h}} Z_{\zeta} {\rm e}^{-m_{\rm bare}T} \left\{ f_1^{\rm stat} + \omega_{\rm kin} f_1^{\rm kin} - \tfrac{1}{3}\omega_{\rm spin} f_1^{\rm spin} \right\} \qquad \text{in HQET}$$

$\Rightarrow c_{\rm A}^{\rm HQET} f_{\delta {\rm A}}^{\rm stat}$ and $c_{\rm V}^{\rm HQET} f_{\delta {\rm A}}^{\rm stat}$ do not appear.

Observables (I)

Define the ratios

$$R_1^{\mathrm{P}} \equiv \ln \frac{f_1(\theta)}{f_1(\theta')} \qquad R_1^{\mathrm{V}} \equiv \ln \frac{k_1(\theta)}{k_1(\theta')} \qquad R_1^{\mathrm{stat}} \equiv \ln \frac{f_1^{\mathrm{stat}}(\theta)}{f_1^{\mathrm{stat}}(\theta')}$$

and the observable

$$\begin{aligned}
\Phi_1 &\equiv \underbrace{\tfrac{1}{4}(R_1^{\mathrm{P}} + 3R_1^{\mathrm{V}})}_{\mathrm{QCD}} - \underbrace{R_1^{\mathrm{stat}}}_{\mathrm{HQET}} \\
&= \omega_{\mathrm{kin}} \underbrace{\left\{ \frac{f_1^{\mathrm{kin}}(\theta)}{f_1^{\mathrm{stat}}(\theta)} - \frac{f_1^{\mathrm{kin}}(\theta')}{f_1^{\mathrm{stat}}(\theta')} \right\}}_{\mathrm{HQET}} \\
&\equiv \omega_{\mathrm{kin}} R_1^{\mathrm{kin}}
\end{aligned}$$

Observables (II)

For the spin averaged mass, we define $f_1^{\rm av} = f_1^{1/4} k_1^{3/4}$ and

$$\begin{aligned}
\Phi_2^{\rm QCD}(L,M) &\equiv L\Gamma_1^{\rm QCD} \equiv -L\widetilde{\partial_T} \ln f_1^{\rm av}\Big|_{T=L/2} \\
&\equiv \frac{L}{2a}\left[\ln f_1^{\rm av}(T+a) - \ln f_1^{\rm av}(T-a)\right]\Big|_{T=L/2} \\
\Phi_2^{\rm HQET}(L,M) &= L[m_{\rm bare} + \Gamma_1^{\rm stat} + \omega_{\rm kin}\Gamma_1^{\rm kin}]
\end{aligned}$$

where

$$\Gamma_1^{\rm stat} = -\widetilde{\partial_T} \ln f_1^{\rm stat} \qquad \Gamma_1^{\rm kin} = -\widetilde{\partial_T}\left[f_1^{\rm kin}/f_1^{\rm stat}\right]$$

Step scaling functions

We define the static step scaling function

$$\sigma_{\mathrm{m}}(u) \quad = \quad 2L \left[\Gamma_1^{\mathrm{stat}}(2L) - \Gamma_1^{\mathrm{stat}}(L) \right] \Big|_{u=\bar{g}^2(L)}$$

Step scaling functions

We define the static step scaling function

$$\sigma_{\mathrm{m}}(u) \quad = \quad 2L\left[\Gamma_1^{\mathrm{stat}}(2L) - \Gamma_1^{\mathrm{stat}}(L)\right]\Big|_{u=\bar{g}^2(L)}$$

and the kinetic step scaling functions

$$\sigma_1^{\mathrm{kin}}(u) \quad = \quad \left.\frac{R_1^{\mathrm{kin}}(2L)}{R_1^{\mathrm{kin}}(L)}\right|_{u=\bar{g}^2(L)}$$

$$\sigma_2^{\mathrm{kin}}(u) \quad = \quad 2L\left.\frac{\Gamma_1^{\mathrm{kin}}(2L) - \Gamma_1^{\mathrm{kin}}(L)}{R_1^{\mathrm{kin}}(L)}\right|_{u=\bar{g}^2(L)}$$

Connection to the large volume

Remember that in infinite volume

$$
\begin{aligned}
m_{\mathrm{B}}^{\mathrm{av}} &= E^{\mathrm{stat}} + m_{\mathrm{bare}} + \omega_{\mathrm{kin}} E^{\mathrm{kin}} \\
&- \quad (\Gamma^{\mathrm{stat}}(L_2) + \omega_{\mathrm{kin}} \Gamma^{\mathrm{kin}}(L_2)) \\
&+ \quad (\Gamma^{\mathrm{stat}}(L_2) + \omega_{\mathrm{kin}} \Gamma^{\mathrm{kin}}(L_2)) \\
&= \left[E^{\mathrm{stat}} - \Gamma^{\mathrm{stat}}(L_2)\right] + \left[\omega_{\mathrm{kin}}(E^{\mathrm{kin}} - \Gamma^{\mathrm{kin}}(L_2))\right] \\
&+ \quad m_{\mathrm{bare}} + \underbrace{\Gamma^{\mathrm{stat}}(L_2)}_{\Gamma^{\mathrm{stat}}(L_1) + \frac{\sigma_{\mathrm{m}}(u_1)}{L_2}} + \underbrace{\omega_{\mathrm{kin}} \Gamma^{\mathrm{kin}}(L_2)}_{\omega_{\mathrm{kin}}(\Gamma^{\mathrm{kin}}(L_1) + \frac{R_1^{\mathrm{kin}}(L_1)}{L_2} \sigma_2^{\mathrm{kin}})}
\end{aligned}
$$

Connection to the large volume

Remember that in infinite volume

$$
\begin{aligned}
m_{\rm B}^{\rm av} &= E^{\rm stat} + m_{\rm bare} + \omega_{\rm kin} E^{\rm kin} \\
&\quad - \left(\Gamma^{\rm stat}(L_2) + \omega_{\rm kin}\Gamma^{\rm kin}(L_2)\right) \\
&\quad + \left(\Gamma^{\rm stat}(L_2) + \omega_{\rm kin}\Gamma^{\rm kin}(L_2)\right) \\
&= \left[E^{\rm stat} - \Gamma^{\rm stat}(L_2)\right] + \left[\omega_{\rm kin}(E^{\rm kin} - \Gamma^{\rm kin}(L_2))\right] \\
&\quad + m_{\rm bare} + \underbrace{\Gamma^{\rm stat}(L_2)}_{\Gamma^{\rm stat}(L_1) + \frac{\sigma_{\rm m}(u_1)}{L_2}} + \underbrace{\omega_{\rm kin}\Gamma^{\rm kin}(L_2)}_{\omega_{\rm kin}(\Gamma^{\rm kin}(L_1) + \frac{R_1^{\rm kin}(L_1)}{L_2}\sigma_2^{\rm kin})}
\end{aligned}
$$

Matching 1 $\quad m_{\rm bare} + \Gamma_1^{\rm stat}(L_1) + \omega_{\rm kin}\Gamma_1^{\rm kin}(L_1) = \frac{1}{L_1}\Phi_2^{\rm QCD}(L_1, M)$

Connection to the large volume

Remember that in infinite volume

$$
\begin{aligned}
m_{\mathrm{B}}^{\mathrm{av}} &= E^{\mathrm{stat}} + m_{\mathrm{bare}} + \omega_{\mathrm{kin}} E^{\mathrm{kin}} \\
&\quad - \left(\Gamma^{\mathrm{stat}}(L_2) + \omega_{\mathrm{kin}} \Gamma^{\mathrm{kin}}(L_2)\right) \\
&\quad + \left(\Gamma^{\mathrm{stat}}(L_2) + \omega_{\mathrm{kin}} \Gamma^{\mathrm{kin}}(L_2)\right) \\
&= \left[E^{\mathrm{stat}} - \Gamma^{\mathrm{stat}}(L_2)\right] + \left[\omega_{\mathrm{kin}}(E^{\mathrm{kin}} - \Gamma^{\mathrm{kin}}(L_2))\right] \\
&\quad + m_{\mathrm{bare}} + \underbrace{\Gamma^{\mathrm{stat}}(L_2)}_{\Gamma^{\mathrm{stat}}(L_1) + \frac{\sigma_{\mathrm{m}}(u_1)}{L_2}} + \underbrace{\omega_{\mathrm{kin}} \Gamma^{\mathrm{kin}}(L_2)}_{\omega_{\mathrm{kin}}(\Gamma^{\mathrm{kin}}(L_1) + \frac{R_1^{\mathrm{kin}}(L_1)}{L_2} \sigma_2^{\mathrm{kin}})}
\end{aligned}
$$

Matching 1 $\quad m_{\mathrm{bare}} + \Gamma_1^{\mathrm{stat}}(L_1) + \omega_{\mathrm{kin}} \Gamma_1^{\mathrm{kin}}(L_1) = \frac{1}{L_1} \Phi_2^{\mathrm{QCD}}(L_1, M)$

Matching 2 $\quad \omega_{\mathrm{kin}} = \frac{\Phi_1(L_1)}{R_1^{\mathrm{kin}}(L_1)} = \frac{\Phi_1^{\mathrm{QCD}}(L_1) + \Phi_1^{\mathrm{stat}}(L_1)}{R_1^{\mathrm{kin}}(L_1)}$

Isolate the $1/m$ term

$$m_{\rm B}^{\rm av}(M) = m_{\rm B}^{\rm stat}(M) + m_{\rm B}^{(1)}(M)$$

Isolate the $1/m$ *term*

$$m_{\rm B}^{\rm av}(M) = m_{\rm B}^{\rm stat}(M) + m_{\rm B}^{(1)}(M)$$

$$m_{\rm B}^{\rm stat}(M) \quad = \quad \underbrace{[E^{\rm stat} - \Gamma_1^{\rm stat}(L_2)]}_{a\to 0 \text{ in HQET}} + \underbrace{\frac{\sigma_m(u_1)}{L_2}}_{a\to 0 \text{ in HQET}} + \underbrace{2\Phi_2^{\rm QCD}(L_1, M)}_{a\to 0 \text{ in QCD}}$$

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$$m_{\rm B}^{(1)}(M) = \underbrace{\frac{\sigma_2^{\rm kin}(u_1)\Phi_1(L_1)}{L_2}}_{a\to 0 \text{ in HQET \& QCD}} + \underbrace{[(E^{\rm kin} - \Gamma_1^{\rm kin}(L_2))\frac{\Phi_1(L_1)}{R_1^{\rm kin}(L_2)}\sigma_1^{\rm kin}(u_1)]}_{a\to 0 \text{ in HQET \& QCD}}$$

Isolate the $1/m$ term

$$m_{\rm B}^{\rm av}(M) = m_{\rm B}^{\rm stat}(M) + m_{\rm B}^{(1)}(M)$$

$$m_{\rm B}^{\rm stat}(M) \quad = \quad \underbrace{[E^{\rm stat} - \Gamma_1^{\rm stat}(L_2)]}_{a\to 0 \text{ in HQET}} + \underbrace{\frac{\sigma_m(u_1)}{L_2}}_{a\to 0 \text{ in HQET}} + \underbrace{2\Phi_2^{\rm QCD}(L_1, M)}_{a\to 0 \text{ in QCD}}$$

$$m_{\rm B}^{(1)}(M) \quad = \quad \underbrace{\frac{\sigma_2^{\rm kin}(u_1)\Phi_1(L_1)}{L_2}}_{a\to 0 \text{ in HQET \& QCD}} \quad \underbrace{+ [(E^{\rm kin} - \Gamma_1^{\rm kin}(L_2))\frac{\Phi_1(L_1)}{R_1^{\rm kin}(L_2)}\sigma_1^{\rm kin}(u_1)]}_{a\to 0 \text{ in HQET \& QCD}}$$

$m_{\rm B}^{(1a)}(M) = \frac{1}{L_2}\sigma_2^{\rm kin}(u_1)\Phi_1(L_1)$ small volume

$m_{\rm B}^{(1b)}(M) = [(E^{\rm kin} - \Gamma_1^{\rm kin}(L_2))\frac{\Phi_1(L_1)}{R_1^{\rm kin}(L_2)}\sigma_1^{\rm kin}(u_1)]$ small & large volume

- We define $M_{\rm b}^{\rm stat}$ as the solution of $m_{\rm B}^{\rm exp} = m_{\rm B}^{\rm stat}(M_{\rm b}^{\rm stat})$

  and the NLO correction $M_{\rm b}^{(1)}$ as $M_{\rm b} = M_{\rm b}^{\rm stat} + M_{\rm b}^{(1)} + O(1/M^2)$

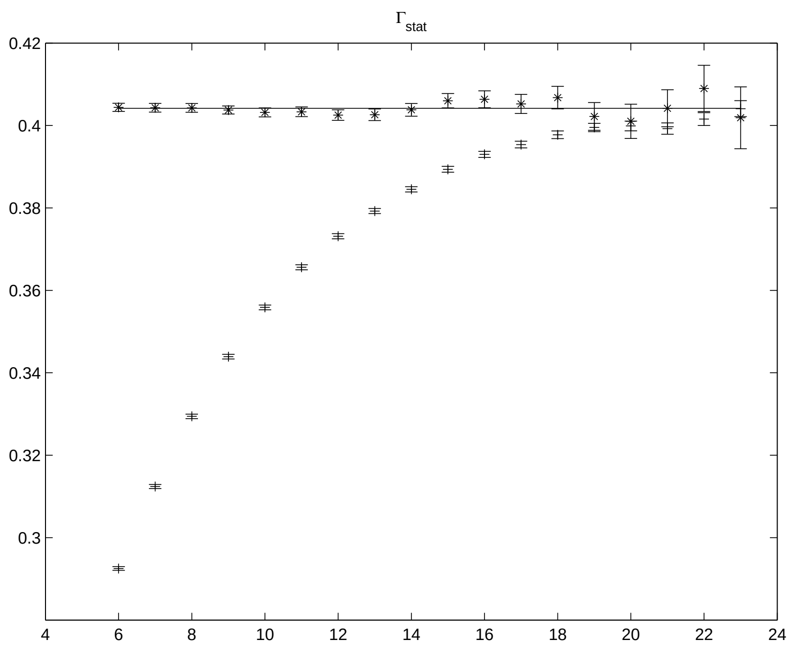
- With the slope $S = \frac{dm_{\rm B}^{\rm stat}}{dM}(M_{\rm b}^{\rm stat})$, the Taylor expansion of the B meson mass around $M_{\rm b}^{\rm stat}$ gives

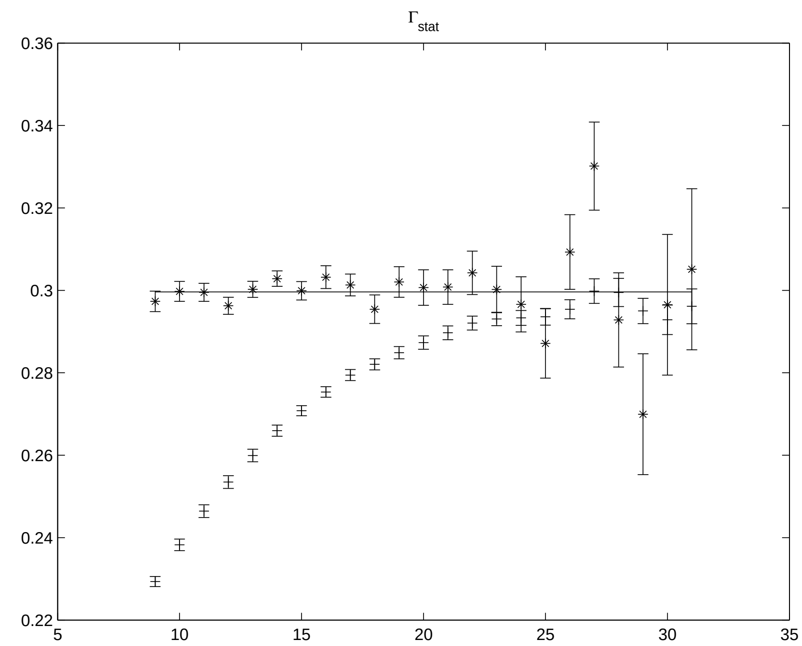
$$m_{\rm B}^{\rm exp} = m_{\rm B}^{\rm stat}(M_{\rm b}^{\rm stat}) + SM_{\rm b}^{(1)} + m_{\rm B}^{(1)}(M_{\rm b}^{\rm stat}) + O(1/M^2)$$

$$\Rightarrow \qquad M_{\rm b}^{(1)} = -\frac{m_{\rm B}^{(1)}(M_{\rm b}^{\rm stat})}{S}$$

Results

- $E^{\rm stat} \equiv \Gamma_1^{\rm stat}(L_\infty)$, and $L_\infty = 2L_2 \simeq 1.5\,\mathrm{fm}$.
- To cancel the power divergences, $E^{\rm stat}$ and $\Gamma_1^{\rm stat}(L_2)$ are computed at the same value of the lattice spacing
- $E^{\rm stat} - \Gamma_1^{\rm stat}(L_2)$ is then computed for two different lattice spacing, and extrapolated to the continuum.

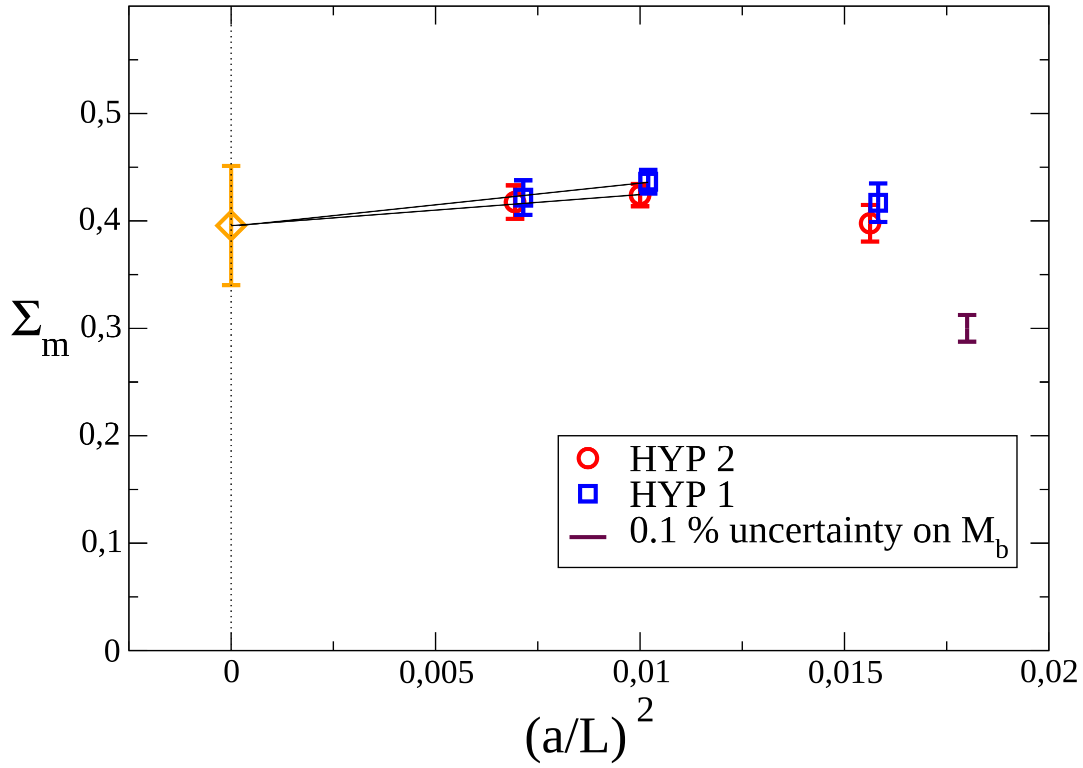




Static step scaling function

$$\sigma_{\mathrm{m}}(u) = \lim_{a/L_1 \to 0} \Sigma_{\mathrm{m}}(u, a/L_1)$$

$$\Sigma_{\mathrm{m}}(u, a/L_1) = 2L_1[\Gamma^{\mathrm{stat}}(2L_1, a) - \Gamma^{\mathrm{stat}}(L_1, a)]\big|_{u=\bar{g}^2(L_1)}$$



Continuum extrapolation of $\Phi_2^{\mathrm{QCD}}(L_1, M)$

From earlier calculations $M_{\mathrm{b}}^{\mathrm{RGI}} L_1 \simeq 12$

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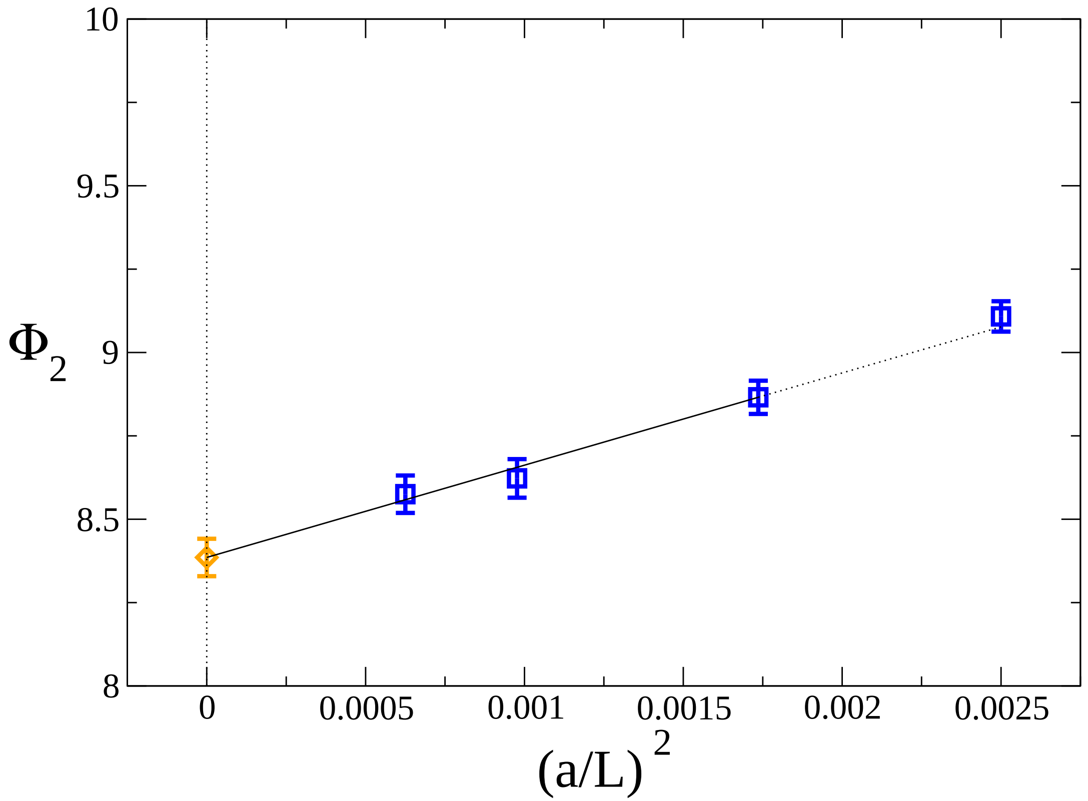
$\Rightarrow$ Simulate 3 values of $z = ML_1 = 10.3, 12, 13.2$, at 4 different betas.

Continuum extrapolation of $\Phi_2^{\mathrm{QCD}}(L_1, M)$

From earlier calculations $M_{\mathrm{b}}^{\mathrm{RGI}} L_1 \simeq 12$

$\Rightarrow$ Simulate 3 values of $z = ML_1 = 10.3, 12, 13.2$, at 4 different betas.

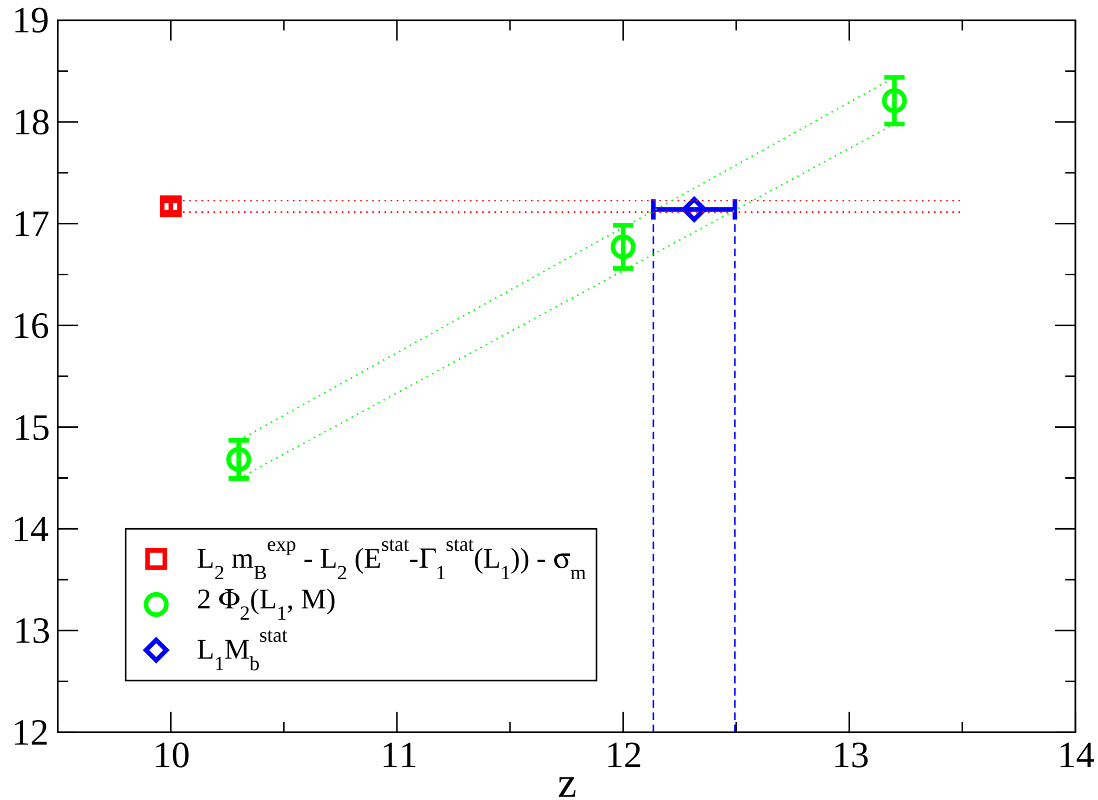
$\Rightarrow$ Continuum extrapolation for each of these masses,
eg for $z = 12$



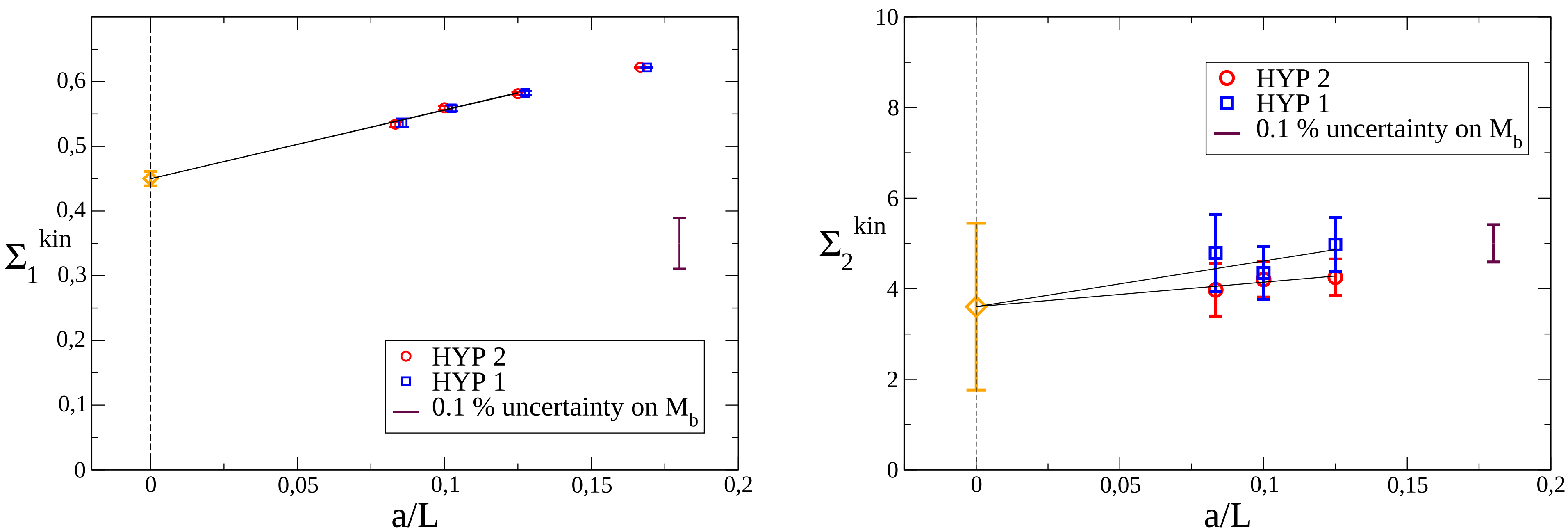
Interpolation

$$L_2 m_{\rm B}^{\rm stat}(M) = L_2[E^{\rm stat} - \Gamma_1^{\rm stat}(L_2)] + \sigma_m(u_1) + 2\Phi_2(L_1, M)$$

We solve $m_{\rm B}^{\rm stat}(M_{\rm b}^{\rm stat}) = m_{\rm B}^{\rm exp} = 5404\,{\rm MeV}$ by a linear interpolation



Kinetic step scaling functions



RGI results

Using $m_{\mathrm{B}}^{\mathrm{exp}} = 5404$ MeV and $r_0 = 0.5\,\mathrm{fm}$ we found (quenched results)

$$r_0 M_{\mathrm{b}}^{\mathrm{stat}} = \begin{cases} 17.18(25) & \text{(HYP2)} \\ 17.15(25) & \text{(HYP1)} \end{cases} \qquad M_{\mathrm{b}}^{\mathrm{stat}} = \begin{cases} 6771(99) \text{ MeV} & \text{(HYP2)} \\ 6757(99) \text{ MeV} & \text{(HYP1)} \end{cases}$$

θ	θ'	$r_0 M_{\mathrm{b}}^{(1\mathrm{a})}$	$r_0 M_{\mathrm{b}}^{(1\mathrm{b})}$
0	0.5	$-0.08(4)$	$-0.14(11)$
0.5	1	$-0.08(4)$	$-0.15(11)$
1	0	$-0.08(4)$	$-0.15(11)$

$$M_{\mathrm{b}}^{(1\mathrm{a})} = -30(15)\ \mathrm{MeV} \qquad M_{\mathrm{b}}^{(1\mathrm{b})} = -56(43)\ \mathrm{MeV}$$

$\overline{\mathrm{MS}}$ *results*

With $\Lambda^{(0)}_{\overline{\mathrm{MS}}} = 238\,\mathrm{MeV}$ [Capitani, Lüscher, Sommer, Wittig 98],

we found in the $\overline{\mathrm{MS}}$ scheme (quenched results)

$$\begin{aligned} m_{\mathrm{b}} &= m_{\mathrm{b}}^{\mathrm{stat}} + m_{\mathrm{b}}^{(1)} \\ m_{\mathrm{b}}^{\mathrm{stat}}(m_{\mathrm{b}}) &= 4.350(64)\,\mathrm{GeV}\,, \quad m_{\mathrm{b}}^{(1)}(m_{\mathrm{b}}) = -0.049(29)\,\mathrm{GeV}\,, \end{aligned}$$

$$\Rightarrow \quad \mathbf{m}_{\mathrm{b}}(\mathbf{m}_{\mathrm{b}}) = \mathbf{4.30(7)}\,\mathrm{GeV}\,.$$

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$$
\begin{aligned}
m_\mathrm{b} &= m_\mathrm{b}^\mathrm{stat} + m_\mathrm{b}^{(1)} \\
m_\mathrm{b}^\mathrm{stat}(m_\mathrm{b}) &= 4.350(64)\,\mathrm{GeV}\,, \quad m_\mathrm{b}^{(1)}(m_\mathrm{b}) = -0.049(29)\,\mathrm{GeV}\,,
\end{aligned}
$$

$$\Rightarrow \quad \mathbf{m}_\mathrm{b}(\mathbf{m}_\mathrm{b}) = \mathbf{4.30(7)}\,\mathrm{GeV}\,.$$

Other results :

$4.41(5)(10)$	[Martinelli & Sachrajda [98]]	NLO matching
$4.30(5)(5)$	[Martinelli & Sachrajda 98],[Lubicz 01]	NNLO matching
$4.12(7)(4)$	[Heitger & Sommer 03]	NP matching
4.38(2)(4)	[Guazzini & Sommer] Preliminary	$\longrightarrow$ Next talk

Summary and outlook

- This method allows for non perturbative determination of physical quantities in NLO of HQET
- We obtain

$$m_{\rm b}^{\rm stat}(m_{\rm b}) = 4.350(64)\,{\rm GeV}\,, \qquad m_{\rm b}^{(1)}(m_{\rm b}) = -0.049(29)\,{\rm GeV}\,.$$

- Compute the spin splitting term ($\omega_{\rm spin}$)
- Compute decay constants of heavy-light mesons
- Unquenched ...