Order α^4 QED Contributions to the Bhabha Scattering Cross Section (Part I)

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Outline

1 Bhabha scattering in pure QED: generalities

2 Two-loop virtual corrections in the $m_e = 0$ approximation

Z. Bern, L. J. Dixon and A. Ghinculov, (2000)

3 Virtual and soft $\mathcal{O}(\alpha^4(N_F = 1))$ corrections with $m_e \neq 0$

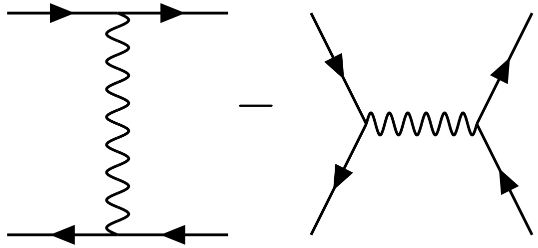
R. Bonciani *et al.*, (2003-04)

4 Virtual photonic vertex corrections and related soft emission diagrams ($m_e \neq 0$)

R. Bonciani and A. F., unpublished

5 Summary & Conclusions

Warning

$\mathcal{M} =$ 

- In this talk we consider the pure QED process (photons + electrons)
- We consider differential cross-sections summed over the spins of the final state particles and averaged over the spin of the initial ones

$$\frac{d\sigma_0(s,t)}{d\Omega} = \frac{\alpha^2}{s}\left\{ \frac{1}{s^2}\left[st + \frac{s^2}{2} + (t-2m^2)^2\right] + \frac{1}{t^2}\left[st + \frac{t^2}{2} + (s-2m^2)^2\right] + \frac{1}{st}\left[(s+t)^2 - 4m^4\right] \right\}$$

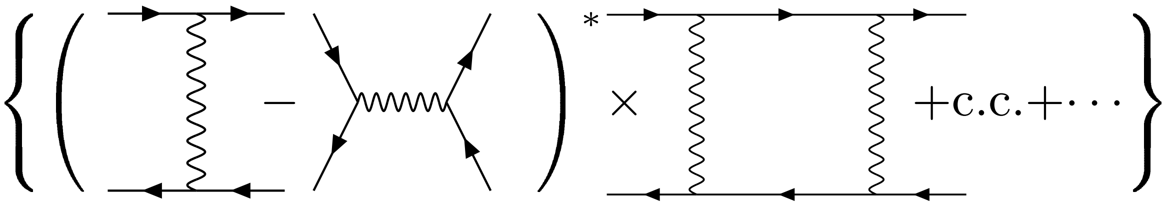
Virtual Corrections to the Cross Section-I

$$\frac{d\sigma(s,t)}{d\Omega} = \frac{d\sigma_0(s,t)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1(s,t)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)^2\frac{d\sigma_2(s,t)}{d\Omega} + \mathcal{O}\left((\alpha/\pi)^3\right)$$

The $\mathcal{O}(\alpha^3)$ virtual corrections (one-loop $\times$ tree-level) are well known (in the full SM), no problem in keeping $m_e \neq 0$

M. Consoli (1979),
M. Böhm, A. Denner, and W. Hollik (1988),
M. Greco (1988),...

$$\left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1^V(s,t)}{d\Omega} = \frac{s}{16}\sum_{\text{spin}}$$

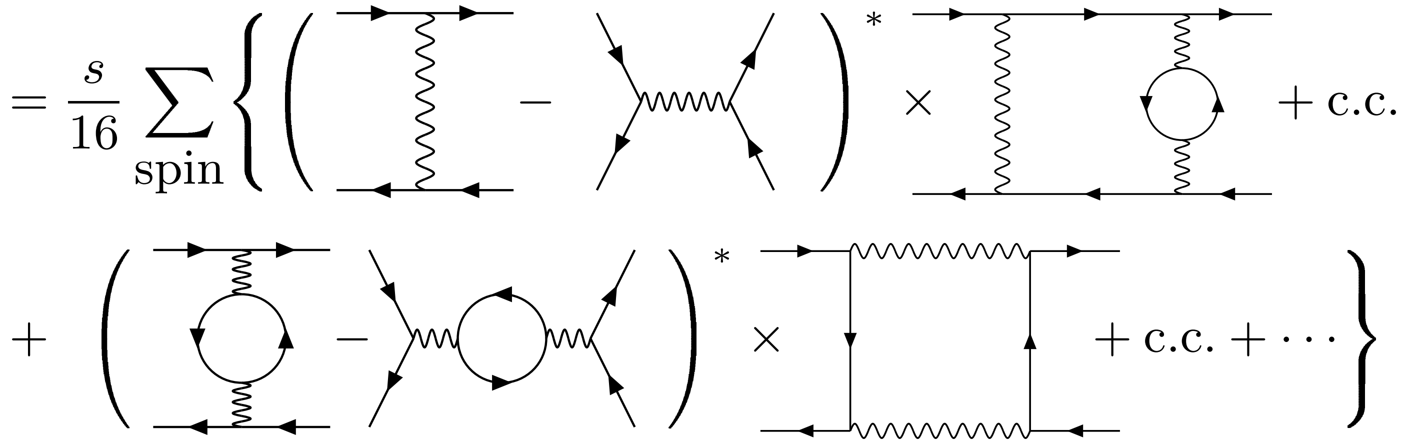


Virtual Corrections to the Cross Section-II

Order α^4 corrections:

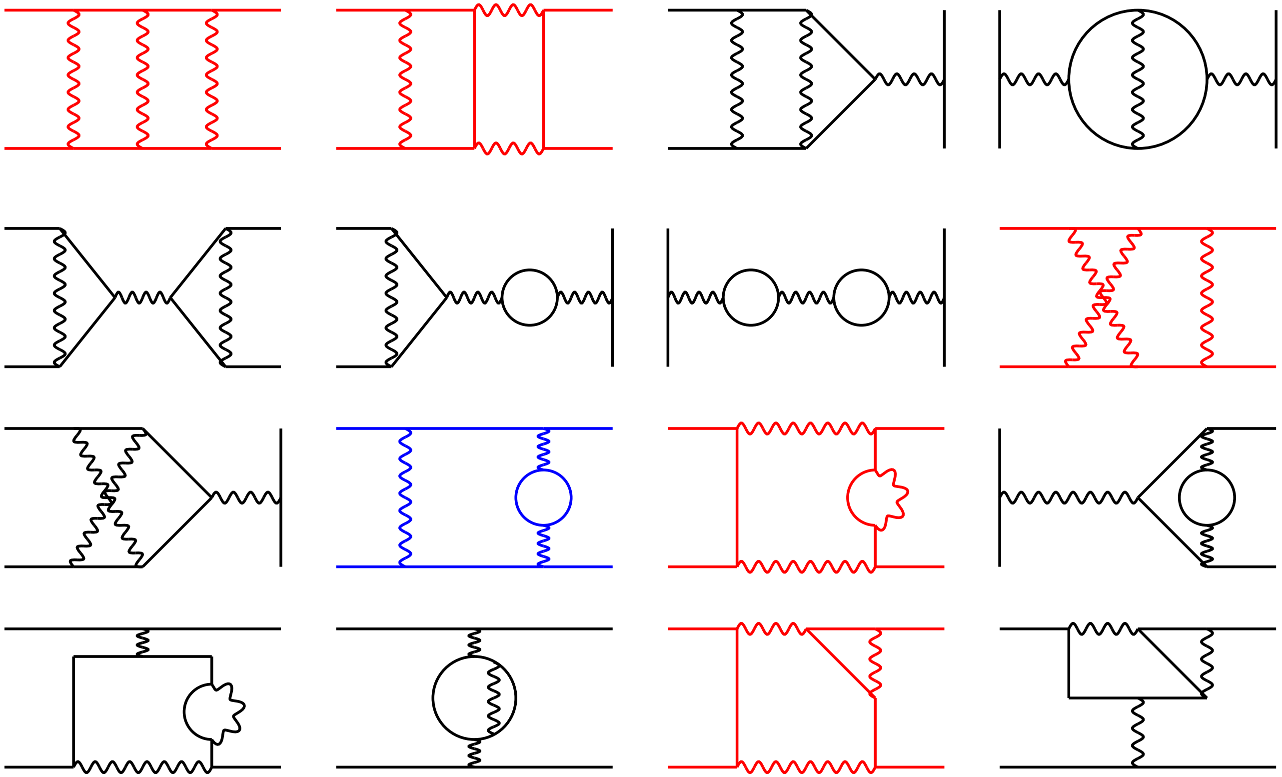
- Contributions from two-loop $\times$ tree-level and one-loop $\times$ one-loop
- The two-loop photonic box diagrams for $m_e \neq 0$ are still missing

$$\left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma_2^V(s,t)}{d\Omega} = \frac{s}{16} \sum_{\text{spin}}$$



2-Loop QED diagrams

suppressing fermion arrows



The Technical Problem: 2-loop boxes

The complexity of the calculation of a Feynman diagram grows with the number of external legs, the number of loops and the number of different scales involved

The calculation of 2L boxes is a roadblock

However

- All the box integrals necessary for the evaluation of the 2L virtual QED corrections in the $m = 0$ approx. are known

  V. Smirnov ('99), B. Tausk ('99), C. Anastasiou, E. Glover, C. Oleari ('00), T. Gehrmann and E. Remiddi ('00), C. Anastasiou, T. gehrmann, C. Oleari, E. Remiddi, and J. B. Tausk ('00)

- For a non-vanishing electron mass the (scalar) planar double boxes have been calculated

  V. Smirnov ('01) V. Smirnov and G. Heinrich ('04)

- The two-loop box with a closed fermion loop and $m_e \neq 0$ is known

  R. Bonciani, A. F., P. Mastrolia, E. Remiddi, and J.J van der Bij ('03)

  M. Czakon, J. Gluza, and T. Riemann ('04)

Two-loop virtual CS in the massless limit

In 2000 Z. Bern, L. Dixon, and A. Ghinkulov calculated the $\mathcal{O}(\alpha^4)$ UV renormalized virtual corrections coming from the two-loop $\times$ tree level interference in the $m_e = 0$ limit

- Both UV and IR divergences are regularized in DIM REG; the all Dirac algebra is kept D dimensional
- result expressible in terms of polylogarithms

$$\mathrm{Li}_n(x) = \sum_{r=1}^{\infty} \frac{x^r}{r^n} = \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(x) \qquad \mathrm{Li}_2 = -\int_0^x \frac{dt}{t} \ln(1-t)$$

- the final result includes IR divergent terms that match the structure predicted by Catani's formula

$\mathcal{O}(\alpha^4)$ Photonic Corrections

- Building on the BDG result and on works A. B. Arbuzov *et al.*, B. Tausk, N. Glover, and J. J. van der Bij ('01) obtained the terms proportional to $L = \ln m_e^2/s$ of the full (virtual + soft) photonic CS (i. e. graphs including a closed electron loop have been neglected)
- Recently A. Penin obtained also the constant terms of the photonic CS in the m_e^2/s expansion (see Penin's talk)

Therefore, in the expansion

$$\frac{d\sigma_2}{d\sigma_0} = \delta_2^{(2)} \ln^2\left(\frac{s}{m_e^2}\right) + \delta_2^{(1)} \ln\left(\frac{s}{m_e^2}\right) + \delta_2^{(0)} + \mathcal{O}\left(\frac{m_e^2}{s}\right)$$

$\delta_2^{(2)}$, $\delta_2^{(1)}$, and $\delta_2^{(0)}$ are known

$\mathcal{O}(\alpha^4(N_F = 1))$ Virtual corrections

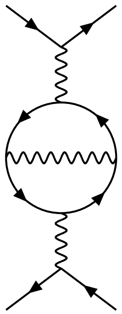
R. Bonciani *et al.*('03-'04)

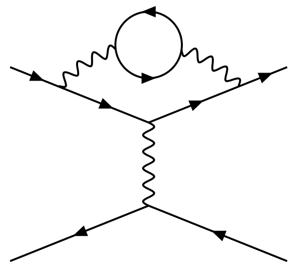
All the two-loop graphs including a closed electron loop can be calculated also keeping $m_e \neq 0$ and without relying on any approximation or expansion

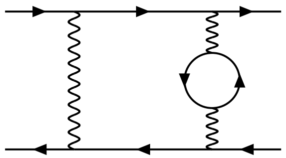
- The relevant integrals can be reduced to combination of a relatively small set of Master Integrals employing the Laporta algorithm
- The MI can be evaluated employing the differential equation method

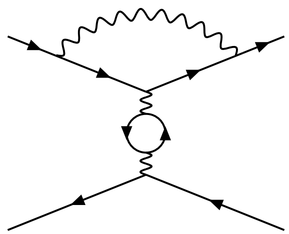
$\longrightarrow$ see Roberto's talk

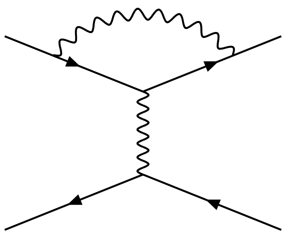
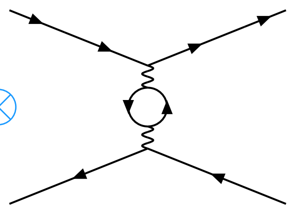
$\mathcal{O}(\alpha^4(N_F = 1))$ Virtual Corrections - Classes of diagrams

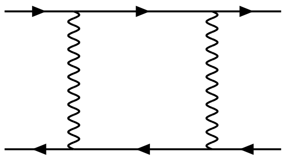
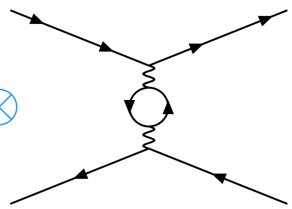
 $\otimes$ Born Amplitude

 $\otimes$ Born Amplitude

 $\otimes$ Born Amplitude

 $\otimes$ Born Amplitude

 $\otimes$ 

 $\otimes$ 

$\mathcal{O}(\alpha^4(N_F = 1))$ virtual corrections-II

In the $\mathcal{O}(\alpha^4(N_F = 1))$ corrections to the CS

- both UV and IR divergences are regularized within the DIM REG scheme
- the UV renormalization is carried out in the on-shell scheme
- the graphs are at first calculated in the non physical region $s < 0$ and then analytically continued to the physical region $s > 4m_e^2$
- the cross section can be expressed in terms of HPLs and 2dHPLs with arguments

$$x = \frac{\sqrt{s} - \sqrt{s - 4m_e^2}}{\sqrt{s} + \sqrt{s - 4m_e^2}} \quad y = \frac{\sqrt{4m_e^2 - t} - \sqrt{-t}}{\sqrt{-t} + \sqrt{4m_e^2 - t}} \quad z = \frac{\sqrt{4m_e^2 - u} - \sqrt{-u}}{\sqrt{-u} + \sqrt{4m_e^2 - u}}$$

$$\frac{(1-z)^2}{z} = \frac{1}{x}\,(x-y)\,(x-1/y)$$

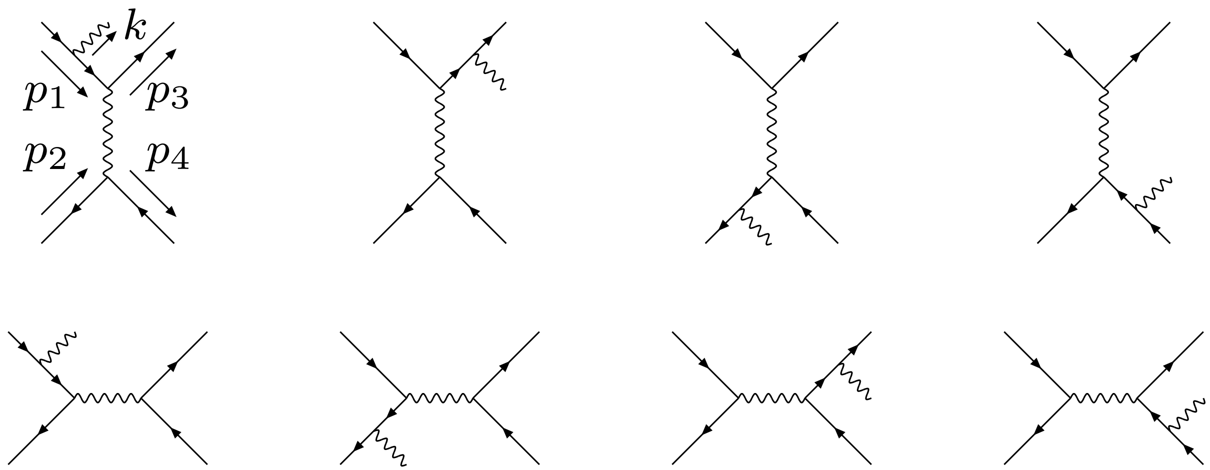
$\mathcal{O}(\alpha^4(N_F = 1))$ Corrections - Soft Photon Emission-I

After UV renormalization, the virtual CS still includes poles in $D-4$, of IR origin, that can be eliminated by adding the contribution of the soft photon emission diagrams

In order to cancel the IR divergent terms in the virtual cross section at $\mathcal{O}(\alpha^3)$ and $\mathcal{O}(\alpha^4(N_F = 1))$ it is sufficient to consider the contribution of the single photon emission graphs

$$e^-(p_1) + e^+(p_2) \longrightarrow e^-(p_3) + e^+(p_4) + \gamma(k) \qquad k_0 < \omega$$

$\mathcal{O}(\alpha^4(N_F = 1))$ Corrections - Soft Photon Emission-II



- The soft emission at order α^3 is obtained by considering the interference among single emission tree level diagrams
- it factorizes with respect to the tree level

$$\left(\frac{\alpha}{\pi}\right) \frac{d\sigma_1^S(s,t,m^2)}{d\Omega} = \left(\frac{\alpha}{\pi}\right) \frac{d\sigma_0^D(s,t,m^2)}{d\Omega} \sum_{i,j=1}^{4} J_{ij}$$

$\Longrightarrow$

$\mathcal{O}(\alpha^4(N_F = 1))$ Corrections - Soft Photon Emission-III

- J_{ij} is an integral of the photon phase space

$$J_{ij} = \epsilon_i \epsilon_j \frac{p_i \cdot p_j}{\Gamma\left(3 - \frac{D}{2}\right) \pi^{(D-4)/2}} \frac{m^{D-4}}{4\pi^2} \int^{\omega} \frac{d^D k}{k_0} \frac{1}{(p_i \cdot k)(p_j \cdot k)}$$

- The integral is IR DIV $\rightarrow$ regularized in DIM REG
- ω is the cut off on the energy of the emitted photon
- J_{ij} depends on E, m_e and on the scalar product $p_i \cdot p_j$ so that it satisfies the symmetry properties

$$J_{ij} = J_{ji} \qquad J_{12} = J_{34} \qquad J_{13} = J_{24} \qquad J_{14} = J_{23}$$

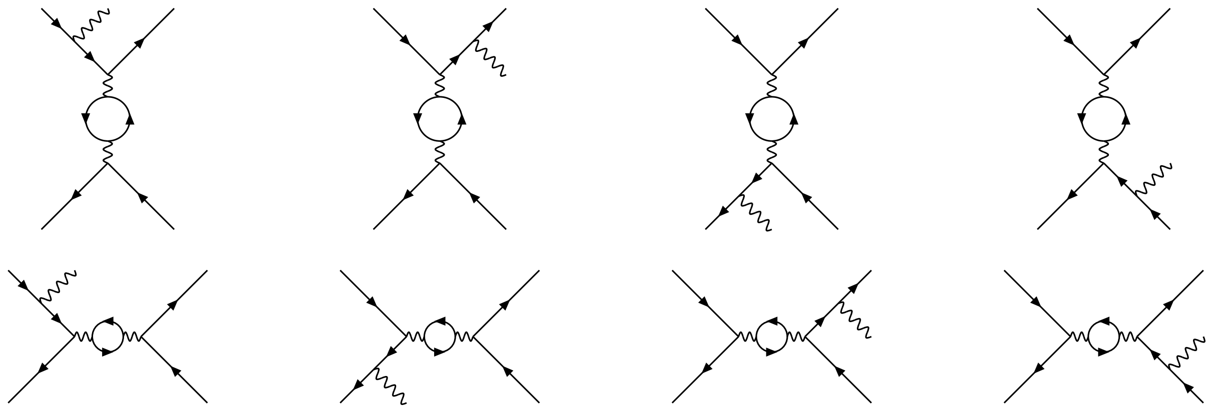
- it is convenient to expand it around $D = 4$ before performing the integration; the result can be expressed in terms of the dimentionless variables x, y, and z $\Longrightarrow$ see Roberto's talk

$\mathcal{O}(\alpha^4(N_F = 1))$ Corrections - Soft Photon Emission-IV

Similarly, at the soft emission cross section at $\mathcal{O}(\alpha^4(N_F = 1))$ is

$$\left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma_1^S(s,t,m^2)}{d\Omega} = \left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma_1^D(s,t,m^2)}{d\Omega} \sum_{i,j=1}^{4} J_{ij}$$

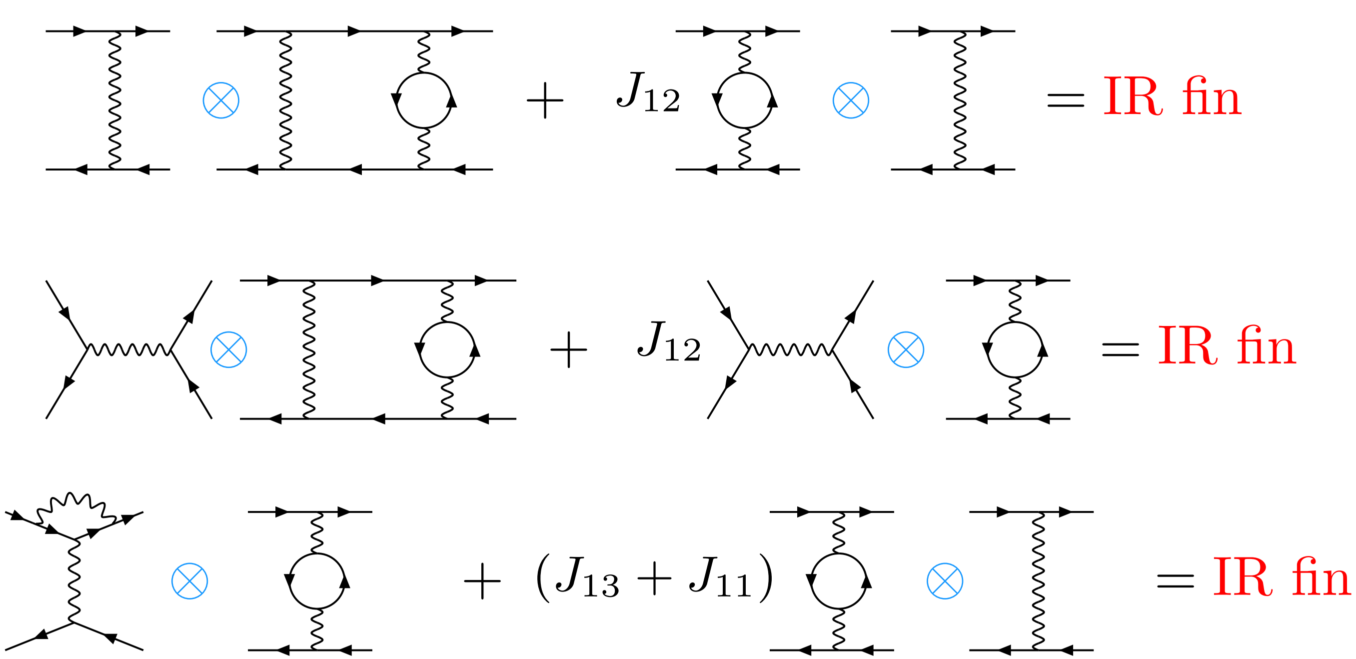
where σ_1^D is the interference of the tree level soft emission with



Remember: σ_1^D is finite (after UV renormalization)

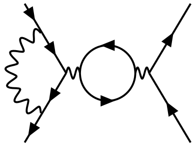
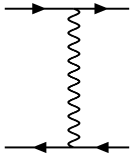
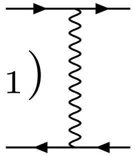
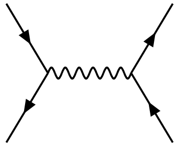
$\mathcal{O}(\alpha^4(N_F = 1))$ Corrections - Soft Photon Emission-V

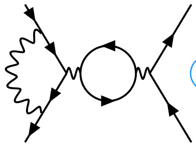
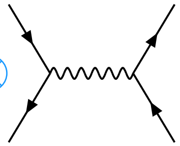
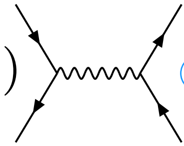
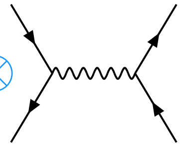
It is possible to understand how the cancellation of the IR poles works from a diagrammatic point of view:

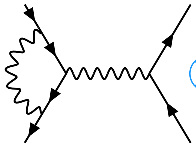
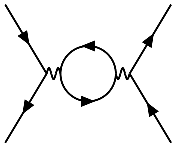
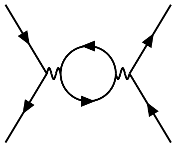
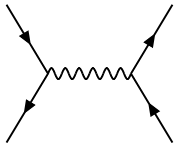


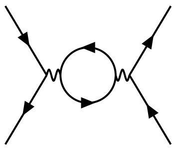
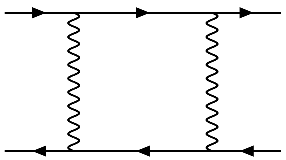
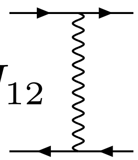
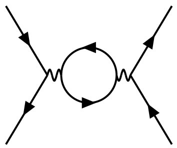
$\mathcal{O}(\alpha^4(N_F = 1))$ Corrections - Soft Photon Emission-VI

In some cases we are left with residual poles proportional to $\zeta(2)$

 $\otimes$  $+ \; (J_{12} + J_{11})$  $\otimes$  = IR pole $\propto \zeta(2)$

 $\otimes$  $+ \; 2\,(J_{12} + J_{11})$  $\otimes$  = IR pole $\propto \zeta(2)$

 $\otimes$  $+ \; (J_{12} + J_{11})$  $\otimes$  = IR pole $\propto \zeta(2)$

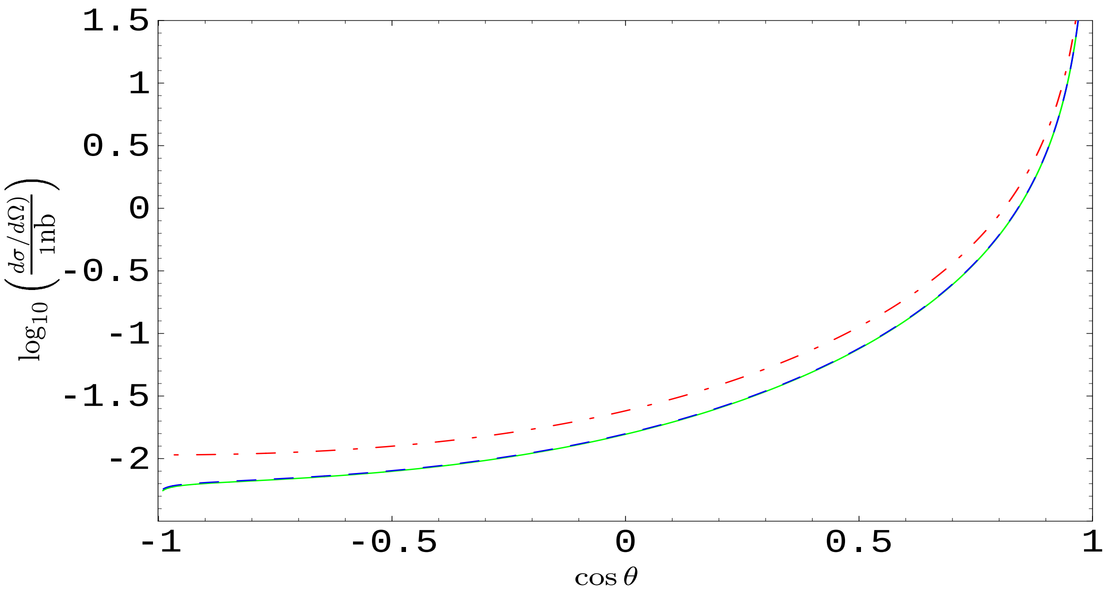
 $\otimes$  $+ \; 2J_{12}$  $\otimes$  = IR pole $\propto \zeta(2)$

The residual pole originates from the analytical continuation of $\ln^2 x$

They cancel among themselves

$\mathcal{O}(\alpha^4(N_F = 1))$ Corrections - Numerical Evaluation of the CS

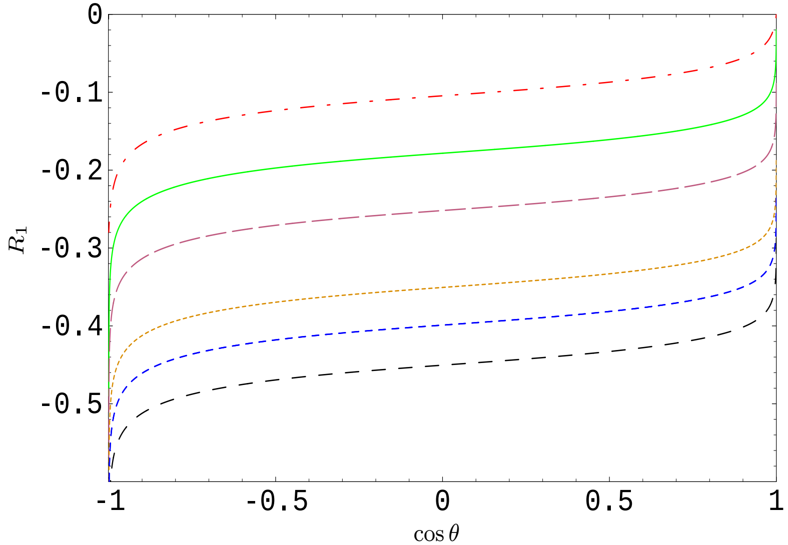
- In order to asses the relative weight of the $\mathcal{O}(\alpha^4(N_F = 1))$ QED corrections to the CS a numerical evaluation is necessary
- For cross-checking purposes we employed two independent codes, one written in `Fortran 77`, the other written in `Mathematica`



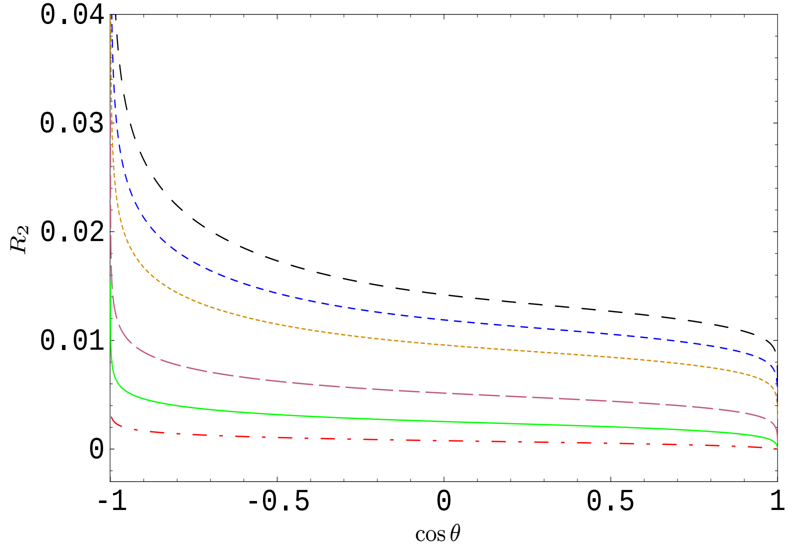
$$E = 22\,\text{GeV} \qquad \omega = 0.1E$$

$\mathcal{O}(\alpha^4(N_F = 1))$ Corrections - Numerical Evaluation of the CS-II

better to look at the ratios of the $\mathcal{O}(\alpha^3)$ and $\mathcal{O}(\alpha^4(N_f = 1))$ CS to the Born CS



$$R_1 = \frac{\alpha}{\pi}\left(\frac{d\sigma_1^T}{d\Omega}\right)\left(\frac{d\sigma_0}{d\Omega}\right)^{-1}$$



$$R_2 = \left(\frac{\alpha}{\pi}\right)^2\left(\frac{d\sigma_2^T}{d\Omega}\right)\left(\frac{d\sigma_0}{d\Omega} + \frac{\alpha}{\pi}\frac{d\sigma_1^T}{d\Omega}\right)^{-1}$$

$E = 10\,\text{MeV}, 100\,\text{MeV}, 1\,\text{GeV}, 22\,\text{GeV}, 100\,\text{GeV}, \text{and } 500\,\text{GeV}$

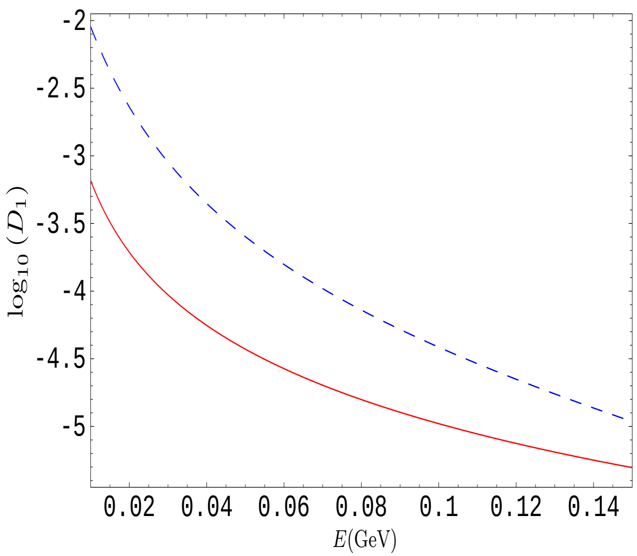
Top to bottom in R_1

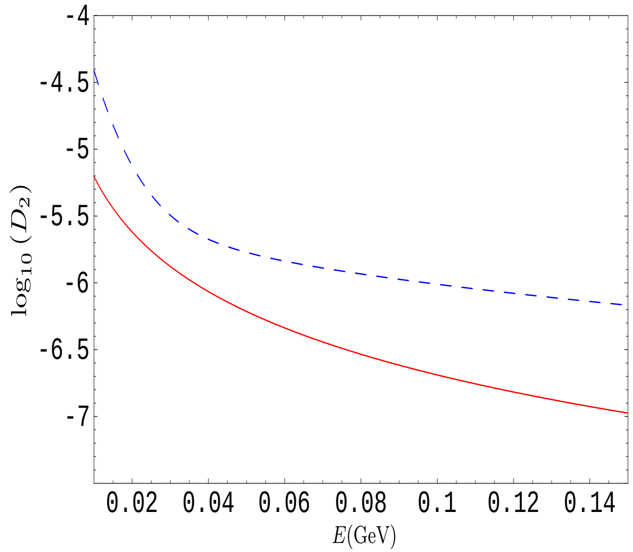
$\omega = 0.1E$

$\mathcal{O}(\alpha^4(N_F = 1))$ Corrections - Expansion for $m_e \to 0$

How well does the expansion in the $m_e \to 0$ limit approximates the exact result?

$$\frac{d\sigma_i^T}{d\Omega} = \left.\frac{d\sigma_i^T}{d\Omega}\right|_L + \mathcal{O}\left(\frac{m_e^2}{s}, \frac{m_e^2}{t}, \frac{m_e^2}{u}\right)$$





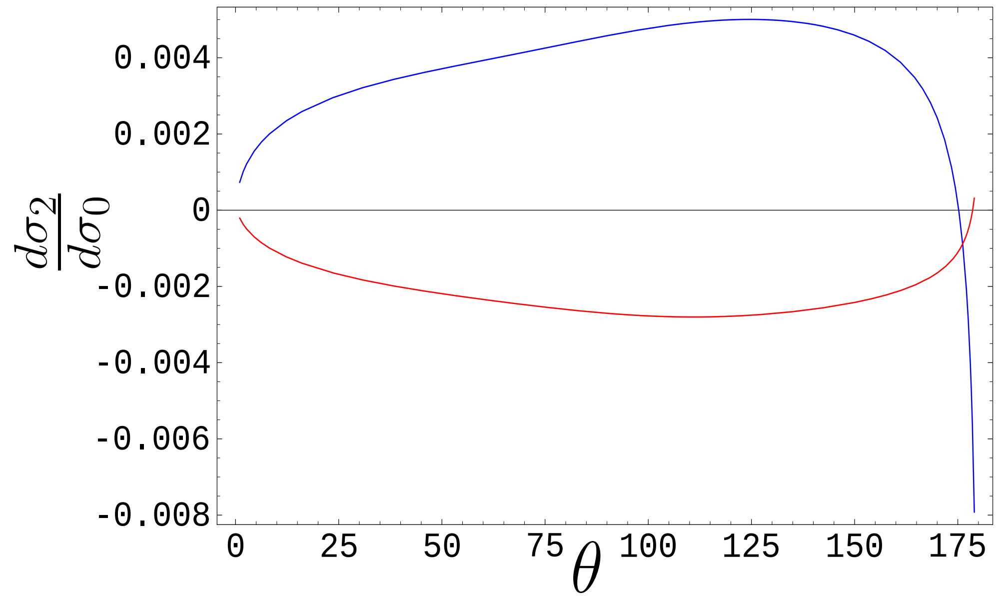
$$D_1 = \frac{\alpha}{\pi}\left(\left.\frac{d\sigma_1^T}{d\Omega}\right|_L - \frac{d\sigma_1^T}{d\Omega}\right)\left(\frac{d\sigma_0}{d\Omega}\right)^{-1}$$

$$D_2 = \left(\frac{\alpha}{\pi}\right)^2\left(\left.\frac{d\sigma_2^T}{d\Omega}\right|_L - \frac{d\sigma_2^T}{d\Omega}\right)\left(\frac{d\sigma_0}{d\Omega} + \frac{\alpha}{\pi}\frac{d\sigma_1^T}{d\Omega}\right)^{-1}$$

dashed line $\to \theta = 90°$, solid line $\to \theta = 5°$- $\omega = 0.1E$

$\mathcal{O}(\alpha^4(N_F = 1))$ Corrections - Comparison with the Photonic CS

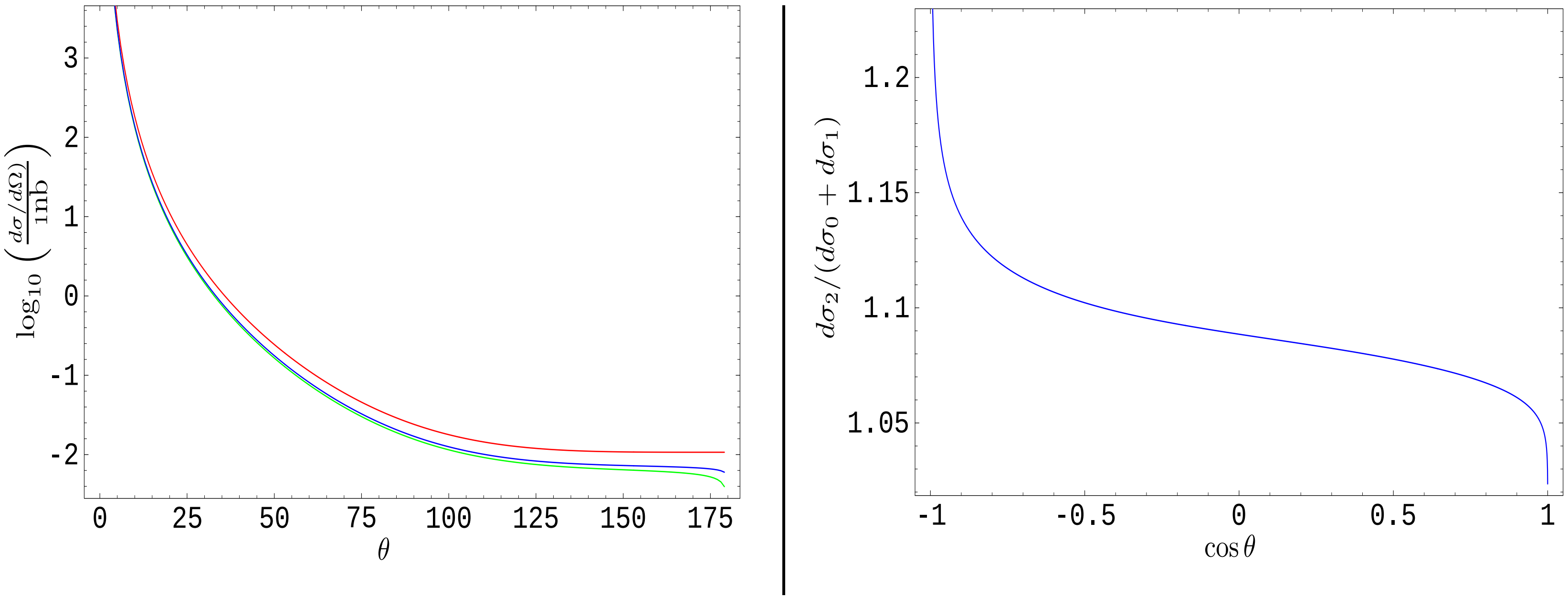
It is interesting to compare the expansion of the $\mathcal{O}(\alpha^4(N_F = 1))$ CS in the $m_e \to 0$ limit with Penin's result for the $\mathcal{O}(\alpha^4)$ photonic CS



keeping terms of $\mathcal{O}\left(\ln^2\left(\frac{s}{m_e^2}\right), \ln\left(\frac{s}{m_e^2}\right), \left(\frac{m_e^2}{s}\right)^0\right)$, setting $E = 0.5\,\text{GeV}$, $\omega = E$

Complete $\mathcal{O}(\alpha^4)$ Corrections in the $m_e \to 0$ limit

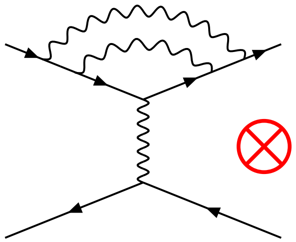
The numerical calculation of the full (photonic + $N_F = 1$ & virtual +soft) $\mathcal{O}(\alpha^4)$ cross section in the $m_e \to 0$ limit has been implemented in a `Mathematica` and a `Fortran77` code



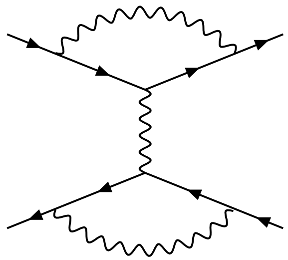
$$E = 22\,\text{GeV} \qquad \omega = 0.1E$$

Vertex Corrections-I

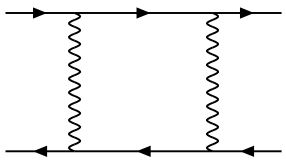
With the same techniques employed in obtaining the $\mathcal{O}(\alpha^4(N_F = 1))$ non-approximated differential CS, it is possible to calculate the contribution of the photonic vertex diagrams (and related soft photon emission) to the CS at order $\mathcal{O}(\alpha^4)$



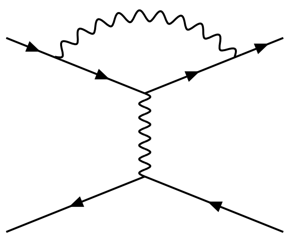
$\otimes$ Born Amplitude

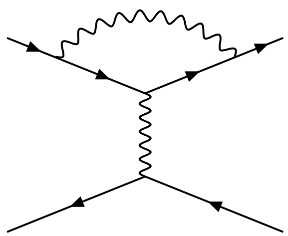


$\otimes$ Born Amplitude

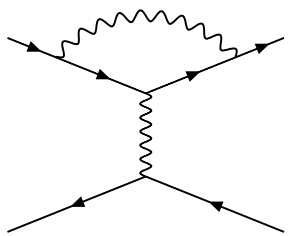


$\otimes$



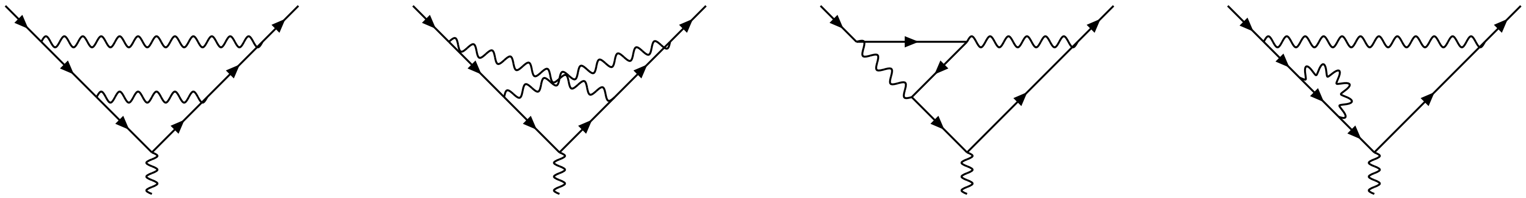


$\otimes$



Vertex Corrections-II

- The two-loop irreducible photonic vertex corrections are gauge independent

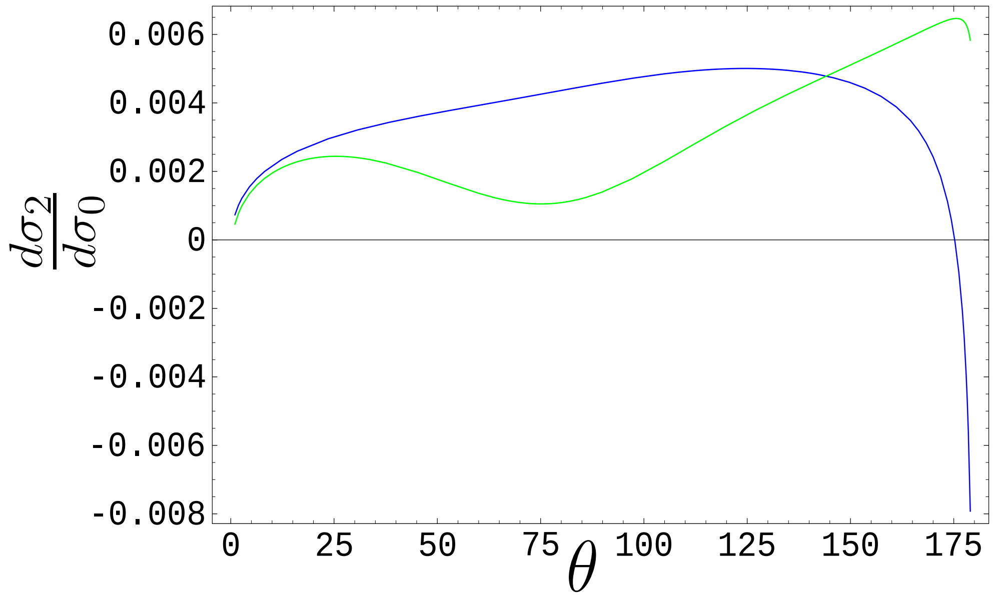


- Their non-approximated ($m_e \neq 0$) expression was obtained by `R. Bonciani, P. Mastrolia, and E.Remiddi ('03)`
- In order to cancel the IR poles it is necessary to add also the contribution of the double photon (soft) emission

$$\frac{d\sigma_2^{(S)}(s,t,m^2)}{d\Omega} = \frac{1}{2}\frac{d\sigma_0^{D}(s,t,m^2)}{d\Omega}\left(\sum_{i,j=1}^{4} J_{ij}\right)^2 + \frac{d\sigma_1^{(V,D)}(s,t,m^2)}{d\Omega}\left(\sum_{i,j=1}^{4} J_{ij}\right)$$

Comparison with Penin's Result

It is possible to compare the expansion of the "vertex" set of corrections with Penin's complete result for the photonic corrections



keeping terms of $\mathcal{O}\left(\ln^2\left(\frac{s}{m_e^2}\right), \ln^2\left(\frac{s}{m_e^2}\right), \left(\frac{m_e^2}{s}\right)^0\right)$, setting $E = 0.5\,\text{GeV}$, $\omega = E$

As expected the vertex correction largely dominates the photonic correction for small angles

Summary & Conclusions-I

Bhabha scattering differential cross section at order $\mathcal{O}(\alpha^4)$

What we have:

- $\surd$ Complete calculation of the virtual corrections in the massless limit
- $\surd$ Complete non-approximated calculation of the $N_F = 1$ corrections (virtual + real)
- $\surd$ Complete calculation of the photonic corrections up to terms of order $\mathcal{O}((m_e^2/s)^0)$

$\Longrightarrow$ Penin

- $\surd$ Complete list of the MIs for the photonic box diagrams

$\Longrightarrow$ Zeuthen group

- $\surd$ Bits and pieces: vertex corrections, residual soft corrections, one-loop box times one-loop box, etc...

Summary & Conclusions-II

- The complete non-approximated ($m_e^2 \neq 0$) two-loop photonic box virtual corrections are still missing
- The expanded result form both the $N_F = 1$ and photonic correction is enough to obtain an accurate cross section for any realistic beam energy, but...
- ... the calculation of the non-approximated two-loop photonic boxes is an extremely difficult technical challenge

Slides about the HPLs

Harmonic Polylogarithms (HPL)

E. Remiddi, J. Vermaseren (1999); E. Remiddi, T. Gehrmann (2001)

Functions of the variable x and a set of indices $\vec{a}$ with weight w;
each index can assume values $1, 0, -1$

$$\mathrm{H}(\mathbf{a};\, \mathrm{x})$$

Definitions: $w = 1$

$$\begin{aligned} H(1;x) &= \int_0^x \frac{dt}{1-t} = -\ln(1-x) \\ H(0;x) &= \ln x \\ H(-1;x) &= \int_0^x \frac{dt}{1+t} = \ln(1+x) \end{aligned}$$

$$\frac{d}{dx} H(a;x) = f(a;x) \quad f(1;x) = \frac{1}{1-x} \quad f(0;x) = \frac{1}{x} \quad f(-1;x) = \frac{1}{1+x}$$

Definitions: $w > 1$

$$\text{if } \vec{a} = 0, 0, \ldots, 0 \ (w \text{ times}) \quad H(\vec{0}_w; x) = \frac{1}{w!} \ln^w x$$

$$\text{else} \quad H(i, \vec{a}; x) = \int_0^x dt\, f(i; t) H(\vec{a}; t)$$

consequences: $\frac{d}{dx} H(i, \vec{a}; x) = f(i; x) H(\vec{a}; t) \quad H(\vec{a} \notin \vec{0}; 0) = 0$

a few examples @ $w = 2$

$$H(0, 1; x) = \int_0^x dt\, f(0; t) H(1; t) = -\int_0^x dt \frac{1}{t} \ln(1 - t) = \mathrm{Li}_2(x)$$

$$H(1, 0; x) = \int_0^x dt\, f(1; t) H(0; t) = \int_0^x dt \frac{1}{1-t} \ln t$$

$$= -\ln x \ln(1 - x) + \mathrm{Li}_2(x)$$

HPLs as a Generalization of the Nielsen's PolyLogs

The HPLs include the Nielsen's PolyLogs

$$S_{n,p(x)} = \frac{(-1)^{n+p-1}}{(n+p)!p!} \int_0^1 \frac{dt}{t} \ln^{n-1} t \ln^p(1-xt) \quad \mathrm{Li}_n(x) = S_{n-1,1}(x)$$

$$\begin{aligned} \mathrm{Li}_n(x) &= H(\vec{0}_{n-1}, 1; x) \\ S_{n,p}(x) &= H(\vec{0}_n, \vec{1}_p; x) \end{aligned}$$

but the HPLs are a larger set of functions: from $w = 4$ one finds things as

$$H(-1,0,0,1;x) = \int_0^x \frac{dt}{1+t} \mathrm{Li}_3(x) \notin \sum \text{Nielsen's PolyLogs}$$

The HPLs Algebra

- Shuffle Algebra:

$$H(\vec{p};x)H(\vec{q};x) = \sum_{\vec{r}=\vec{p}\uplus\vec{q}} H(\vec{r};x)$$

some examples

$$\begin{aligned} H(a;x)H(b;x) &= H(a,b;x)+H(b,a;x) \\ H(a;x)H(b,c;x) &= H(a,b,c;x)+H(b,a,c;x)+H(b,c,a;x) \end{aligned}$$

- Product Ids:

$$\begin{aligned} H(m_1,\ldots,m_q;x) &= H(m_1;x)H(m_2,\ldots,m_q;x) \\ &- H(m_2,m_1;x)H(m_3,\ldots,m_q;x) \\ &+ \cdots + (-1)^{q+1}H(m_q,\ldots,m_1;x) \end{aligned}$$

2-dimensional Harmonic Polylogarithms (2dHPL)

E. Remiddi, T. Gehrmann (2000)

As for the HPLs, they are obtained by repeated integration over a new set of factors depending on a second variable.

$$f(-y;x) = \frac{1}{x+y} \qquad f(-1/y;x) = \frac{1}{x+1/y}$$

$$G(i,\vec{a};x) = \int_0^x dt\, f(i;t)\, G(\vec{a};t)$$

a few examples:

$$G(-y;x) = \int_0^x \frac{dz}{z+y} = \ln\left(1+\frac{x}{y}\right) \qquad G(-1/y;x) = \int_0^x \frac{dz}{z+1/y} = \ln(1+xy)$$

$$G(-y,0;x) = \ln x \ln\left(1+\frac{x}{y}\right) + \mathrm{Li}_2\left(-\frac{x}{y}\right)$$

The 2dHPLs share the properties of the HPLs

Up to $w = 3$ (our case) the 2dHPLs can be expressed in terms of

$\ln, \mathrm{Li}_2, \mathrm{Li}_3, S_{1,2}$

The analytic properties of both HPLs & 2dHPLs are know

Codes for their numerical evaluation are available

E. Remiddi, T. Gehrmann (2001-2002)