Order α^4 QED Contributions to the Bhabha Scattering Cross-Section

(Part II)

Roberto BONCIANI

Fakultät für Mathematik und Physik
Albert-Ludwigs-Universität Freiburg,
D-79104 Freiburg, Germany.

In collaboration with:

A. Ferroglia, P. Mastrolia, E. Remiddi, and J. J. van der Bij

Karlsruhe, April 22, 2005

Plan of the Talk

- The Reduction to Master Integrals (MIs)
  - The Laporta Algorithm
  - The Differential Equations Method (Harmonic Polylogarithms)
- Differential Cross-Section
  - Virtual Contribution
  - Real Soft-Photon Emission and IR Singularities
- Summary

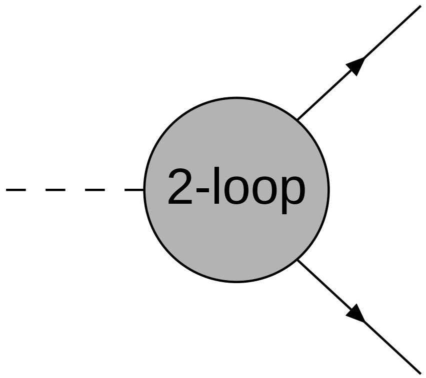
Reduction to the MIs and their Evaluation

Calculation of the Feynman Diagrams

- Decomposition of the amplitude in scalar Form Factors and Projection operators. Direct evaluation of the matrix elements, traces and expression of the observables in terms of dimensionally regularized scalar integrals

- Reduction of the scalar quantities to a combination of a small set of independent scalar integrals (the Master Integrals) via the "Laporta Algorithm"
- Evaluation of the Master Integrals with the Differential Equations Method and expression of the solution in a suitable basis of functions. In several cases it turned out that the basis of HPLs is a good choice

Scalar Integrals



$$\sim \int d^D k_1 \, d^D k_2 \, \frac{(k_i^\mu, p_i^\mu, \gamma^\mu)}{D_1 \cdots D_t}$$

The coefficients of the decomposition of $\int d^D k_1 \, d^D k_2$ on the basis of known tensors are combinations of the following scalar integrals:

$$I \;=\; \int d^D k_1 \, d^D k_2 \, \frac{S_1^{n_1} \cdots S_q^{n_q}}{D_1^{m_1} \cdots D_t^{m_t}} \qquad \in I_{t,r,s}$$

$$r = \sum_i (m_i - 1) \qquad s = \sum_i n_i$$

The METHOD

1 **Reduction of the complete set of diagrams to the Master Integrals (MIs)**

Construction of the relations among dim-regularized integrals. Algebraic method

- Integration by Parts Identities
- Lorentz Invariance
- Symmetry relations.

2 **Calculation of the MIs**

Construction of a system of first order linear differential equations for the MIs. Solution order-by-order in ϵ

S. Laporta and E. Remiddi, *Phys. Lett.* **B379** (1996) 283;
S. Laporta, *Int. J. Mod. Phys.* **A 15** (2000) 5087 (hep-ph/0102033).

Integration by Parts Identities (IBPs)

It is possible to derive a class of identities for $I_{t,r,s}$ from the following relation:

$$\int d^D k \, \frac{\partial}{\partial k^\mu} \, J^\mu(k,p) \; = \; 0$$

where $J^\mu(k,p)$ is a combination of p_i^μ, S_i, and D_i.

For a two-loop amplitude we find 8-10 Equations for each input amplitude:

$$\int d^D k_1 d^D k_2 \, \frac{\partial}{\partial k_1^\mu} \, \left[v^\mu \frac{S_1^{n_1} \cdots S_q^{n_q}}{D_1^{m_1} \cdots D_t^{m_t}} \right] \; = \; 0$$

$$\int d^D k_1 d^D k_2 \, \frac{\partial}{\partial k_2^\mu} \, \left[v^\mu \frac{S_1^{n_1} \cdots S_q^{n_q}}{D_1^{m_1} \cdots D_t^{m_t}} \right] \; = \; 0$$

where $v^\mu = k_1^\mu$, k_2^μ, p_i^μ, with $i = 2$ or 3.

F. V. Tkachov, *Phys. Lett.* **B100** (1981) 65;
K. G. Chetyrkin and F. V. Tkachov, *Nucl. Phys.* **B192** (1981) 159.

Lorentz-Invariance Identities

The integrals $I_{t,r,s}$ are Lorentz scalars $I(p_i + \delta p_i) = I(p_i), \; \delta p_i^\mu = \epsilon_\nu^\mu p_i^\nu$. We have:

$$I(p_i + \delta p_i) \simeq I(p_i) + \sum_i \delta p_i^\mu \frac{\partial I}{\partial p_i^\mu} = I(p_i) + \epsilon_\nu^\mu \left[\sum_i p_i^\nu \frac{\partial}{\partial p_i^\mu} \right] I$$

that implies:

$$\epsilon_\nu^\mu \left[\sum_i p_i^\nu \frac{\partial}{\partial p_i^\mu} \right] I = 0$$

because ϵ_ν^μ is an anti-symmetric tensor, we have:

$$\sum_i \left\{ p_i^\nu \frac{\partial}{\partial p_i^\mu} - p_i^\mu \frac{\partial}{\partial p_i^\nu} \right\} I = 0$$

that can be contracted with an anti-symmetric combination of $p_i p_j$ finding 1-3 Equations for each input amplitude.

T. Gehrmann and E. Remiddi, *Nucl. Phys.* **B580** (2000) 485.

In particular

- For a 3-point function:

$$\left[p_1\cdot p_2\left(p_1^\mu\frac{\partial}{\partial p_1^\mu}-p_2^\mu\frac{\partial}{\partial p_2^\mu}\right)+p_2^2\,p_1^\mu\frac{\partial}{\partial p_2^\mu}-p_1^2\,p_2^\mu\frac{\partial}{\partial p_1^\mu}\right]I(p_i) = 0$$

- For a 4-point function:

$$\left(p_1^\mu p_2^\nu - p_1^\nu p_2^\mu\right)\sum_n\left[p_n^\nu\frac{\partial}{\partial p_n^\mu}-p_n^\mu\frac{\partial}{\partial p_n^\nu}\right]I(p_i) = 0$$

$$\left(p_1^\mu p_3^\nu - p_1^\nu p_3^\mu\right)\sum_n\left[p_n^\nu\frac{\partial}{\partial p_n^\mu}-p_n^\mu\frac{\partial}{\partial p_n^\nu}\right]I(p_i) = 0$$

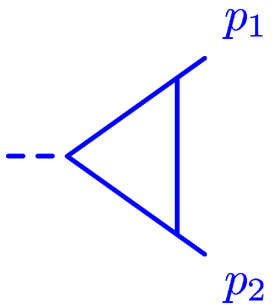
$$\left(p_2^\mu p_3^\nu - p_2^\nu p_3^\mu\right)\sum_n\left[p_n^\nu\frac{\partial}{\partial p_n^\mu}-p_n^\mu\frac{\partial}{\partial p_n^\nu}\right]I(p_i) = 0$$

General-Symmetry Relations

Explicit symmetries are also used in the reduction process.

For example:

$$I_{t,r,s} =$$



$$= \int \mathfrak{D}^D k_1 \, \mathfrak{D}^D k_2 \frac{S_1^{n_1} \cdots S_q^{n_q}}{D_1^{m_1} \cdots D_t^{m_t}} = F(Q^2, a)$$

where $Q^2 = (p_1 + p_2)^2$ and $p_1^2 = p_2^2 = -a$.

Invariance under transformations that leave Q^2 and a (and therefore I) unchanged gives non-trivial relations.

Suitable mass distributions give also identity relations:

Up-Down (UD)

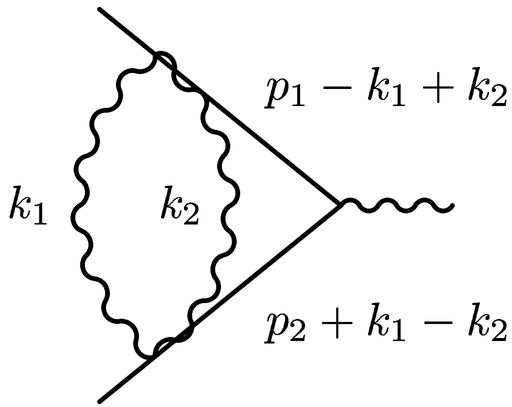
$$\begin{cases} p_1 & \leftrightarrow & p_2 \\ k_1 & \rightarrow & k_1' = \alpha_1 k_1 + \alpha_2 k_2 + ... \\ k_2 & \rightarrow & k_2' = \beta_1 k_1 + \beta_2 k_2 + ... \end{cases}$$

$$UD\,[F] - F \;=\; 0$$

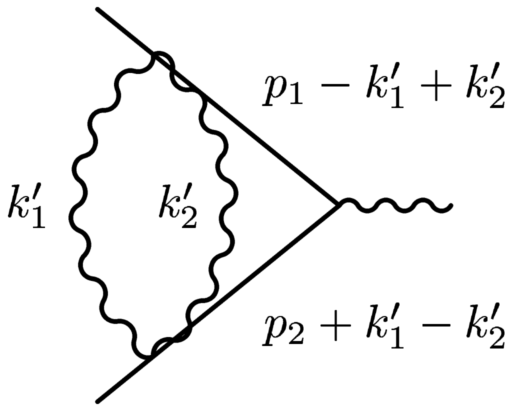
Right-Left (RL)

$$\begin{cases} p_1 & \leftrightarrow & -p_3 \\ p_2 & \leftrightarrow & -p_4 \\ k_1 & \rightarrow & k_1' = \alpha_1 k_1 + \alpha_2 k_2 + ... \\ k_2 & \rightarrow & k_2' = \beta_1 k_1 + \beta_2 k_2 + ... \end{cases}$$

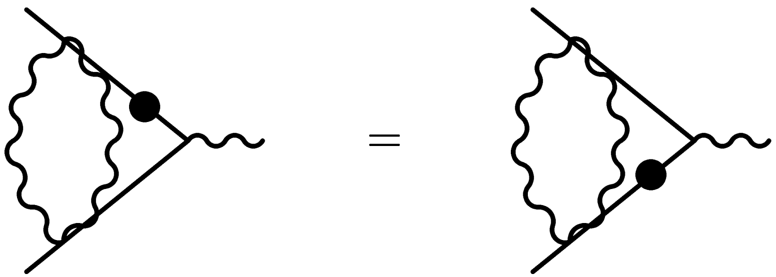
$$RL\,[F] - F \;=\; 0$$

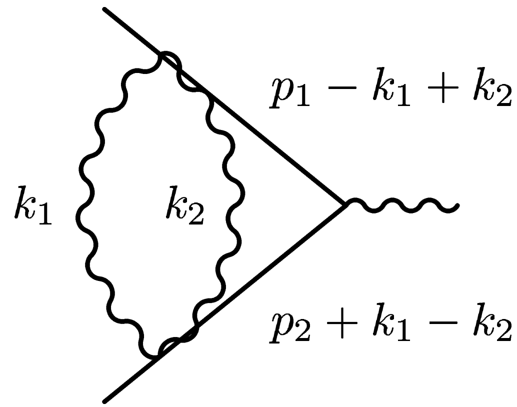


$$\xrightarrow[\begin{cases} k_1 = -k_1' \\ k_2 = -k_2' \end{cases}]{p_1 \leftrightarrow p_2}$$

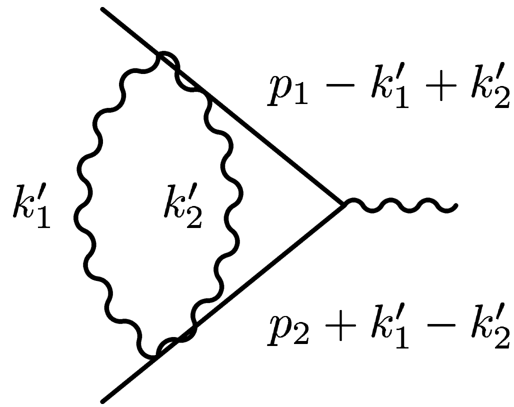


... gives for example:

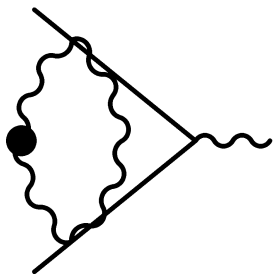




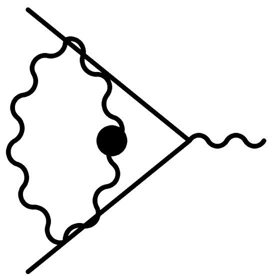
$$\longrightarrow \quad \begin{cases} k_1 = -k_2' \\ k_2 = -k_1' \end{cases}$$



... gives for example



$=$



Identity-Relations

- Integration by Parts Identities (IBPs):

$$\int d^D k_1 d^D k_2 \, \frac{\partial}{\partial k^\mu_{1,2}} \left[v^\mu \frac{S_1^{n_1} \cdots S_q^{n_q}}{D_1^{m_1} \cdots D_t^{m_t}} \right] = 0$$

where $v^\mu = k^\mu_1,\, k^\mu_2,\, p^\mu_i$, with $i = 2$ or 3.

8-10 EQUATIONS
for each Amplitude

F. V. Tkachov, *Phys. Lett.* **B100** (1981) 65;
K. G. Chetyrkin and F. V. Tkachov, *Nucl. Phys.* **B192** (1981) 159.

- Lorentz-invariance Identity (LI):

$$\left[p_1 \cdot p_2 \left(p^\mu_1 \frac{\partial}{\partial p^\mu_1} - p^\mu_2 \frac{\partial}{\partial p^\mu_2} \right) + p_2^2 \, p^\mu_1 \frac{\partial}{\partial p^\mu_2} - p_1^2 \, p^\mu_2 \frac{\partial}{\partial p^\mu_1} \right] I(p_i) = 0$$

1-3 EQUAT.
for Ampl.

T. Gehrmann and E. Remiddi, *Nucl. Phys.* **B580** (2000) 485.

- Explicit-Symmetry Relations (Sym):

Up-Down (UD) $\Longrightarrow \begin{cases} p_1 & \leftrightarrow & p_2 \\ k_1 & \rightarrow & k'_1 = \alpha_1 k_1 + \alpha_2 k_2 + \ldots \\ k_2 & \rightarrow & k'_2 = \beta_1 k_1 + \beta_2 k_2 + \ldots \end{cases}$

1-2 EQUATs.
for Ampl.

Equations vs. Unknown Amplitudes

- For each $I_{t,r,s}$ we have

$$N_{ID} = N_{IBPs} + N_{LI} + N_{Sym}$$

  equations that involve different amplitudes of the same topology t, but with different r $(r-1,\ r,\ r+1)$ and s $(s-1,\ s,\ s+1)$. Note that $r-1$ can bring to subtopologies.

- Let us take into account the number of unknown amplitudes. For a given t, $r = \sum_i (m_i - 1)$ $(r \geq 0)$ and $s = \sum_i n_i$ $(s \geq 0)$:

$$N\left[I_{t,r,s}\right] = \binom{r+t-1}{t-1}\binom{6-t+s}{6-t}$$

NOTE that for a given topology t, it exists (r_0,s_0) such that for $r > r_0$ and $s > s_0$:

$$N_{ID} \cdot N\left[I_{t,r,s}\right] > N\left[I_{t,r+1,s+1}\right] + N\left[I_{t,r+1,s}\right] + \ldots$$

Example: 5-denominator topology.

$$t = 5\,, \quad N_{sp} = 7 - t = 2$$

$s \Rightarrow$ / $d \Downarrow$	0	1	2
6	10 *Eqs* 18 *Ampl*	30 *Eqs* 36 *Ampl*	60 *Eqs* 60 *Ampl*
7	60 *Eqs* 63 *Ampl*	180 *Eqs* 126 *Ampl*	360 *Eqs* 210 *Ampl*
8	210 *Eqs* 168 *Ampl*	630 *Eqs* 336 *Ampl*	1260 *Eqs* 560 *Ampl*

Linear System and Solution

- For a given topology, the number of Equations grows faster than the number of Unknown Amplitudes

1 It is possible to generate a number in principle ARBITRARY of equations, overconstraining the system and solving it by the "brute force" (algorithm)

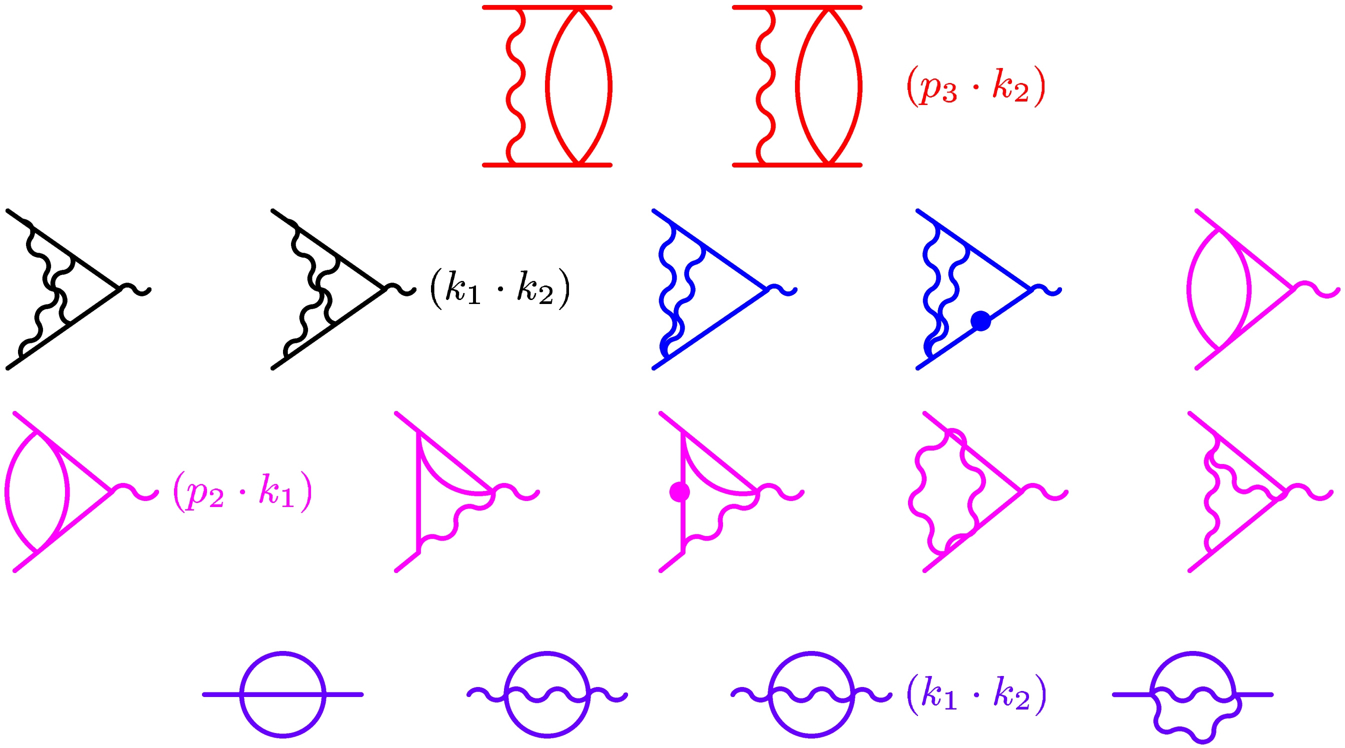
2 The equations are not all independent!
Two hypothesis can occur:

- The system is effectively overconstrained. We obtain $I_{t,r,s}$ as a combination of sub-topologies $I_{t-1,?,?}$, $I_{t-2,?,?}$ ...
- The system is not reducible. We obtain all the integrals of the class $I_{t,r,s}$ as a combination of an hopefully small number of MI's (and sub-topologies)

Reducible

MI's

Two-Loop Master Integrals



R. B., P. Mastrolia and E. Remiddi, *Nucl. Phys.* **B661** (2003) 289.
R. B., A. Ferroglia, P. Mastrolia, E. Remiddi, and J. van der Bij, *Nucl. Phys.* **B681** (2004) 261.
M. Czakon, J. Gluza and T. Riemann, *Nucl. Phys. Proc. Suppl.* **135** (2004) 83.

Differential Equations for the MI's

For a given topology, when the system of identities is not reducible, we have a small number of MI's. In the case of three-point functions:

$$F_i(Q^2, p_1^2, p_2^2) = \int d^D k_1 d^D k_2 \frac{S_1^{n_1} \cdots S_q^{n_q}}{D_1^{m_1} \cdots D_t^{m_t}}$$

Using all the identity-relations (IBP's, LI, Sym) we can construct the following system of first-order linear differential equations:

$$\frac{\partial F_i}{\partial Q^2} = \sum_j h_j(Q^2, m^2)\, F_j + \Omega_{1i}$$

where $i, j = 1, ..., N_{MIs}$.

Ω | This term involves integrals of the class $I_{t-1,r,s}$ (sub-topologies) to be considered KNOWN

A. V. Kotikov, *Phys. Lett.* **B254** (1991) 158; *Phys. Lett.* **B259** (1991) 314; *Phys. Lett.* **B267** (1991) 123.
E. Remiddi, *Nuovo Cim.* **110A** (1997) 1435 (hep-th/9711188).

If we consider four-point functions we have:

$$F_i(P^2, Q^2, p_1^2, p_2^2, p_3^2, p_4^2) = \int d^D k_1 d^D k_2 \frac{S_1^{n_1} \cdots S_q^{n_q}}{D_1^{m_1} \cdots D_t^{m_t}}$$

Using all the identity-relations (IBP's, LI, Sym) we can construct the following systems of first-order linear differential equations:

$$\frac{\partial F_i}{\partial P^2} = \sum_j h_j^{(P)}(P^2, Q^2, m^2)\, F_j + \Omega_{1i}$$

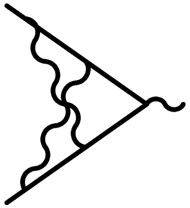
$$\frac{\partial F_i}{\partial Q^2} = \sum_j h_j^{(Q)}(P^2, Q^2, m^2)\, F_j + \Omega_{2i}$$

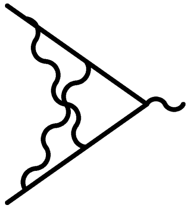
Usually one of the two systems if sufficient for the calculation of the MIs.

$\Omega_{1,2}$ | These terms involve integrals of the class $I_{t-1,r,s}$ (sub-topologies) to be considered KNOWN

Differential Equations for the Crossed Vertex Diagram

The reduction process → to 2 MI's. We choose:

$F_1(\epsilon, a, Q^2) =$ 

$F_2(\epsilon, a, Q^2) =$  $(k_1 \cdot k_2)$

The system of first-order linear differential equations is:

$$\frac{dF_1}{dQ^2} = -(1+2\epsilon)\left[\frac{1}{Q^2} + \frac{(1-2\epsilon)}{(Q^2+4a)}\right] F_1 - \frac{2\epsilon}{a}\left[\frac{1}{Q^2} - \frac{(1-2\epsilon)}{(Q^2+4a)}\right] F_2 + \Omega^{(1)}$$

$$\frac{dF_2}{dQ^2} = \epsilon\, F_1 - \frac{(1-2\epsilon)}{2}\left[\frac{1}{Q^2} + \frac{1}{(Q^2+4a)}\right] F_2 + \Omega^{(2)}$$

Where $\Omega^{(i)}$ are combinations of simpler MI's.

Solution as a Laurent series in ϵ

- We look for a solution expanded in Laurent series of ϵ:

$$F_1(\epsilon, a, Q^2) = \sum_{i=-2}^{0} \epsilon^i F_i^{(1)}(a, Q^2) + \mathcal{O}(\epsilon) \qquad F_2(\epsilon, a, Q^2) = \sum_{i=-2}^{0} \epsilon^i F_i^{(2)}(a, Q^2) + \mathcal{O}(\epsilon)$$

- The homogeneous system at $\epsilon = 0$ $(D = 4)$ decouples:

$$\frac{d f_1(a,y)}{dy} = -\left[\frac{1}{y} + \frac{1}{(y+4a)}\right] f_1(a,y)$$

$$\frac{d f_2(a,y)}{dy} = -\frac{1}{2}\left[\frac{1}{y} + \frac{1}{(y+4a)}\right] f_2(a,y)$$

- The solution of the homogeneous system at $\epsilon = 0$:

$$f_1(a,y) = \frac{k_1}{y(y+4a)} \qquad f_2(a,y) = \frac{k_2}{\sqrt{y(y+4a)}}$$

Euler's Method (variation of constants)

By means of the Euler' method of the variation of the constants k_1 and k_2, we find, order by order in ϵ, the solution of the non-homogeneous system:

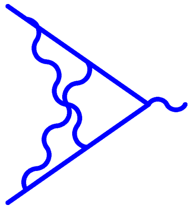
$$
\begin{aligned}
F_i^{(1)}(a,Q^2) &= \frac{1}{Q^2(Q^2+4a)}\Bigg\{\int^{Q^2} dy\, y(y+4a)\Bigg[-2\left(\frac{1}{y}+\frac{1}{(y+4a)}\right)F_{i-1}^{(1)}(a,y) \\
&\quad + \frac{4}{(y+4a)}F_{i-2}^{(1)}(a,y) - \frac{2}{a}\left(\frac{1}{y}-\frac{1}{(y+4a)}\right)F_{i-1}^{(2)}(a,y) \\
&\quad - \frac{4}{a(y+4a)}F_{i-2}^{(2)}(a,y) + \Omega_i^{(1)}(a,y)\Bigg] + k_i^{(1)}\Bigg\} \\
F_i^{(2)}(a,Q^2) &= \frac{1}{\sqrt{Q^2(Q^2+4a)}}\Bigg\{\int^{Q^2} dy\, \sqrt{y(y+4a)}\Bigg[F_{i-1}^{(1)}(a,y) \\
&\quad + \left(\frac{1}{y}-\frac{1}{(y+4a)}\right)F_{i-1}^{(2)}(a,y) + \Omega_i^{(2)}(a,y)\Bigg] + k_i^{(2)}\Bigg\}
\end{aligned}
$$

For the determination of $k_i^{(1)}$ and $k_i^{(2)}$ we have to impose initial conditions.

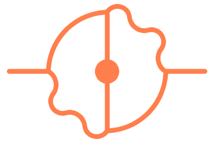
Initial conditions

The vertex topology is not singular at $Q^2 = 0$ (the only singularity is the threshold $Q^2 = -4a$). We can perform the limit directly in the integral getting for F_1:

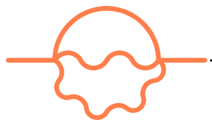
$$F_1(\epsilon, a, Q^2 = 0) = \lim_{Q^2 \to 0}$$



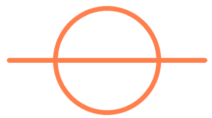
$$=$$



$$= -\frac{3\epsilon(2-3\epsilon)(1-3\epsilon)}{4a^3(1-4\epsilon)(1+2\epsilon)}$$



$$+ \frac{3(2-3\epsilon)(1-3\epsilon)}{64a^3\epsilon}$$

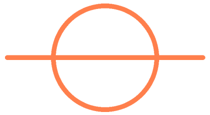


$$+ \frac{(1-\epsilon)^2(9+3\epsilon-160\epsilon^2-196\epsilon^3)}{64a^4\epsilon(1-2\epsilon)(1+2\epsilon)} T^2(\epsilon, a)$$

For F_2 we can perform the limit directly (like for F_1) or we can use the first differential equation, multiply by Q^2 and take the limit $Q^2 \to 0$. We get:

$$F_2(\epsilon, a, Q^2 = 0) = \frac{(2-3\epsilon)(1-3\epsilon)}{8a^2\epsilon}$$

$$+ \frac{(2-3\epsilon)(1-3\epsilon)}{32a^2\epsilon^2}$$



$$+ \frac{(1-\epsilon)^2(3-15\epsilon+16\epsilon^2)}{32a^3\epsilon^2(1-2\epsilon)} T^2(\epsilon, a)$$

NOTE: Regularity at $Q^2 = 0$ imply the ABSENCE of terms

$$\frac{1}{(Q^2)^\alpha} \quad \text{and} \quad (Q^2)^\beta \ln^\gamma(Q^2) \qquad (\alpha, \beta, \gamma > 0)$$

from the expansion of the solution around this point. Therefore we can simply "tune" $k_i^{(1)}$ and $k_i^{(2)}$ such that these terms disappear.

Change of variable and HPL's

It turns out to be a very convenient choice to change the variable Q^2 in x defined as follows:

$$x = \frac{\sqrt{Q^2+4a} - \sqrt{Q^2}}{\sqrt{Q^2+4a} + \sqrt{Q^2}}$$

with which

$$Q^2 = a\frac{(1-x)^2}{x} \qquad (Q^2+4a) = a\frac{(1+x)^2}{x}$$

In terms of x the solutions of the homogeneous system are:

$$\begin{aligned} f_1(a,x) &= -\frac{k_1}{4}\left[\frac{1}{(1-x)} + \frac{1}{(1+x)} - \frac{1}{(1-x)^2} - \frac{1}{(1+x)^2}\right] \\ f_2(a,x) &= \frac{k_2}{2}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] \end{aligned}$$

With this choice the basis for the calculation is constituted by HPL's of x:

$$F_i^{(1)} = \int^x dt \left\{\frac{1}{t}\,;\,\frac{1}{(1-t)}\,;\,\frac{1}{(1+t)}\right\}\left\{F_{i-1}^{(1)}(a,x), F_{i-1}^{(2)}(a,x), \Omega_i^{(1)}(a,x)\right\}$$

Harmonic Polylogarithms (HPLs)

- Weight $= 1$

$$H(0,x) = \ln x \quad H(-1,x) = \int_0^x \frac{dt}{1+t} = \ln(1+x) \quad H(1,x) = \int_0^x \frac{dt}{1-t} = -\ln(1-x)$$

- Weight > 1

If $\vec{a} = \vec{0}$ we define $H(\vec{0},x) = \frac{1}{\omega!}\ln^{\omega} x$. If $\vec{a} \neq \vec{0}$:

$$H(\vec{a},x) = \int_0^x dt\, f(a_1,x)\, H(\vec{a}_{\omega-1},t) \qquad \frac{d}{dx}H(\vec{a},x) = f(a_1,x)\, H(\vec{a}_{\omega-1},x)$$

- The Algebra: $\omega_{\vec{a}} \times \omega_{\vec{b}} = \omega_{\vec{a}} \times \omega_{\vec{b}}$

The product of HPL's of the same argument x and weights a and b is a combination of HPL's of x and weight $a+b$

$$H(\vec{a},x)\, H(\vec{b},x) = \sum_{\vec{c}=\vec{a}\uplus\vec{b}} H(\vec{c},x)$$

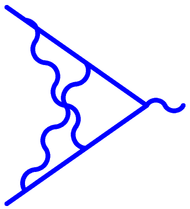
- Integration by Parts

$$H(m_1,...,m_q,x) = H(m_1,x)H(m_2,...,m_q,x) - ... + (-1)^{q+1}H(m_q,...,m_1,x)$$

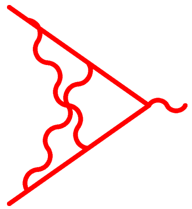
- Connection with Nielsen's polylog: $S_{n,p}(x) = H(\vec{0}_n,\vec{1}_p,x) \quad Li_n(x)=H(\vec{0}_{n-1},1,x)$

E. Remiddi and J. A. M. Vermaseren, *Int. J. Mod. Phys.* **A15** (2000) 725.
T. Gehrmann and E. Remiddi, *Comput. Phys. Commun.* **141** (2001) 296.

Solution for the MI's of the Crossed



$$= \left(\frac{\mu^2}{a}\right)^{2\epsilon} \sum_{i=-1}^{0} \epsilon^i R_i + \mathcal{O}(\epsilon)$$



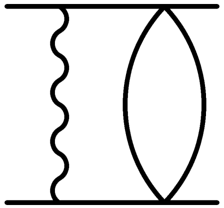
$$(k_1 \cdot k_2) = \left(\frac{\mu^2}{a}\right)^{2\epsilon} S_0 + \mathcal{O}(\epsilon)$$

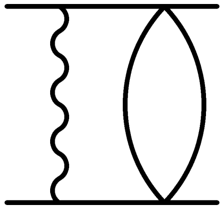
where:

$$\begin{aligned}
a^2 R_{-1} &= -\frac{1}{4}\left[\frac{1}{(1-x)} - \frac{1}{(1-x)^2} + \frac{1}{(1+x)} - \frac{1}{(1+x)^2}\right]\left[\zeta(3)+\zeta(2)H(0,x) + 2H(0,0,0,x)\right.\\
&\quad \left.+2H(0,1,0,x) - 2H(0,-1,0,x)\right]\\
a^2 R_0 &= -\frac{1}{4}\left[\frac{1}{(1-x)} - \frac{1}{(1-x)^2} + \frac{1}{(1+x)} - \frac{1}{(1+x)^2}\right]\left[\frac{37\zeta^2(2)}{10} + H(0,x) - 4H(-1,x) + \zeta(3)H(1,x)\right.\\
&\quad -2\zeta(2)H(0,0,x) - 4\zeta(2)H(-1,0,x) - 2\zeta(2)H(0,-1,x) - 2\zeta(2)H(0,1,x) + 4\zeta(2)H(1,0,x)\\
&\quad +12H(0,0,0,0,x) + 8H(-1,0,-1,0,x) - 8H(-1,0,0,0,x) - 8H(-1,0,1,0,x)\\
&\quad +20H(0,-1,-1,0,x) - 16H(0,-1,0,0,x) - 12H(0,-1,1,0,x) - 24H(0,0,-1,0,x)\\
&\quad -16H(0,0,1,0,x) - 12H(0,1,-1,0,x) + 8H(0,1,0,0,x) + 4H(0,1,1,0,x)\\
&\quad \left.-8H(1,0,-1,0,x) + 8H(1,0,0,0,x) + 8H(1,0,1,0,x)\right]\\
a S_0 &= \left[\frac{1}{(1+x)} - \frac{1}{(1-x)}\right]\left\{\frac{\zeta^2(2)}{10} - \zeta(3)H(0,x) + \zeta(2)(2H(1,0,x) + 3H(0,-1,x))\right.\\
&\quad \left.+\frac{1}{2}H(0,0,0,0,x) + H(0,-1,0,0,x) + H(0,0,-1,0,x) + H(0,1,0,0,x) + 2H(1,0,0,0,x)\right\}
\end{aligned}$$

Differential Equations for a Box Diagram

The reduction process → to 2 MI's. We choose:

$F_1(\epsilon, a, P^2, Q^2) =$ 

$F_2(\epsilon, a, P^2, Q^2) =$  $(p_3 \cdot k_2)$

The system of first-order linear differential equations (in P^2) is decoupled:

$$\frac{dF_1}{dP^2} = -\frac{1}{2}\left[\frac{1}{P^2} + \frac{(1+2\epsilon)}{(P^2+4a)} - \frac{2\epsilon}{(P^2+Q^2+4a)}\right] F_1 + \Omega^{(1)}$$

$$\frac{dF_2}{dP^2} = -\frac{1}{2}\left[\frac{1}{P^2} + \frac{(1+2\epsilon)}{(P^2+4a)} - \frac{2\epsilon}{(P^2+Q^2+4a)}\right] F_2 + \Omega^{(2)}$$

Where $\Omega^{(i)}$ are combinations of simpler MI's.

Solution as a Laurent series in ϵ

- We look for a solution expanded in Laurent series of ϵ:

$$F_1(\epsilon, a, P^2, Q^2) = \sum_{i=-2}^{0} \epsilon^i F_i^{(1)}(a, P^2, Q^2) + \mathcal{O}(\epsilon)$$

- The homogeneous equation at $\epsilon = 0$ ($D = 4$) is:

$$\frac{d f_1(a,x,y)}{dx} = -\frac{1}{2}\left[\frac{1}{x} + \frac{1}{(x+4a)}\right] f_1(a,x,y)$$

- The solution of the homogeneous equation at $\epsilon = 0$:

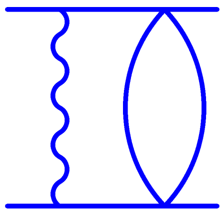
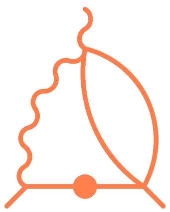
$$f_1(a,x,y) = \frac{k_1}{x(x+4a)}$$

Euler's Method (variation of constants)

By means of the Euler' method of the variation of the constant k_1, we find, order by order in ϵ, the solution of the non-homogeneous equation:

$$F_i^{(1)}(a,Q^2) = \frac{1}{\sqrt{P^2(P^2+4a)}}\left\{\int^{P^2} dx\,\sqrt{x(x+4a)}\times \times\left[\frac{1}{2}\left(\frac{1}{x+4a}-\frac{1}{x+y+4a}\right)F_{i-1}^{(1)}(a,y)+\Omega_i^{(1)}(a,x,y)\right] + k_i^{(1)}\right\}$$

For the determination of $k_i^{(1)}$ we have to impose initial conditions:

$$F_1(\epsilon,a,P^2=0,Q^2) = \lim_{P^2\to 0}$$  $=$ 

Change of variable and HPL's

It turns out to be a very convenient choice to change the variables P^2 and Q^2 in x and y defined as follows:

$$x = \frac{\sqrt{P^2+4a} - \sqrt{P^2}}{\sqrt{P^2+4a} + \sqrt{P^2}} \qquad y = \frac{\sqrt{Q^2+4a} - \sqrt{Q^2}}{\sqrt{Q^2+4a} + \sqrt{Q^2}}$$

with which

$$P^2 = a\frac{(1-x)^2}{x} \quad (P^2+4a) = a\frac{(1+x)^2}{x} \quad Q^2 = a\frac{(1-y)^2}{y} \quad (Q^2+4a) = a\frac{(1+y)^2}{y}$$

$$(P^2+Q^2+4a) = a\frac{1}{x}(x+y)\left(x+\frac{1}{y}\right)$$

With this choice the basis for the calculation is constituted by 2d-HPL's of x, y:

$$F_i^{(1)} = \int^x dt \left\{\frac{1}{t}\, ;\, \frac{1}{(1\pm t)}\, ;\, \frac{1}{(t+y)}\, ;\, \frac{1}{(t+1/y)}\right\} \left\{F_{i-1}^{(1)}(a,x,y), \Omega_i^{(1)}(a,x,y)\right\}$$

2d Harmonic Polylogarithms (2d-HPLs)

- Weight $= 1$

$$G(0,x) = \ln x \quad G(\mp 1,x) = \int_0^x \frac{dt}{1 \pm t} = \pm \ln(1 \pm x) \quad G(-y,x) = \int_0^x \frac{dt}{t+y} = \ln(1 + x/y)$$

$$G(-1/y,x) = \int_0^x \frac{dt}{t+1/y} = \ln(1+xy)$$

- Weight > 1

$\vec{a} = \{-1,0,1,-y,-1/y\}$. If $\vec{a} = \vec{0}$ we define $H(\vec{0},x) = \frac{1}{\omega!}\ln^{\omega} x$. If $\vec{a} \neq \vec{0}$:

$$H(\vec{a},x) = \int_0^x dt\, f(a_1,x)\, H(\vec{a}_{\omega-1},t) \qquad \frac{d}{dx} H(\vec{a},x) = f(a_1,x)\, H(\vec{a}_{\omega-1},x)$$

- The Algebra and the other properties of the HPLs are maintained

- Connection with Nielsen's polylog:

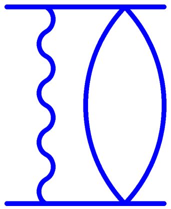
For example:

$$G(-y,0;x) = \ln x \ln\left(1 + \frac{x}{y}\right) + \mathrm{Li}_2\left(-\frac{x}{y}\right)$$

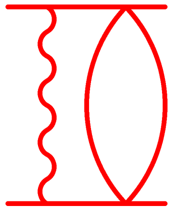
$$G(-1/y,0,0;x) = \frac{\ln^2 x}{2} \ln(1+xy) + \ln x \mathrm{Li}_2(-xy) - \mathrm{Li}_3(-xy)$$

T. Gehrmann and E. Remiddi, *Nucl. Phys.* **B601** (2001) 248.
T. Gehrmann and E. Remiddi, *Comput. Phys. Commun.* **144** (2002) 200.

Solution for the MI's of the Box



$$= \sum_{i=-2}^{0} \epsilon^i R_i + \mathcal{O}(\epsilon)$$



$$(p_3 \cdot k_2) \;\; = \sum_{i=-2}^{0} \epsilon^i S_i + \mathcal{O}(\epsilon)$$

where:

$$a R_{-2} = \frac{1}{8}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] H(0;x)$$

$$a R_{-1} = \frac{1}{16}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] \Bigg\{ \zeta(2) - \left[2 - \left(1 - \frac{2}{(1-y)}\right) H(0;y)\right] H(0;x) - H(0,0;x) + 2H(-1,0;x) \Bigg\}$$

$$\begin{aligned} a R_{0} = -\frac{1}{16}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] \Bigg\{ & \zeta(2) + \zeta(3) - 2H(0;x) - \zeta(2) H(-1;x) - H(0,0;x) + 2H(-1,0;x) - H(0,0,0;x) + H(-1,0,0;x) \\ & + H(0,0;y)\, H(0;x) - 2H(-1,-1,0;x) + H(0,-1,0;x) - \frac{1}{2}\left[1 - \frac{2}{(1-y)}\right] \Big[4\zeta(2) H(0;y) + H(0,0,0;y) + \Big(\zeta(2) - 2H(0;y) \\ & - 4H(0,0;y) - 2H(1,0;y) + 6H(-1,0;y)\Big) H(0;x) + \Big(3\zeta(2) + H(0,0;y)\Big)\Big(G(-y;x) - G(-1/y;x)\Big) + 2H(0;y) H(-1,0;x) \\ & - H(0;y)\Big(G(-y,0;x) + G(-1/y,0;x)\Big) + G(-y,0,0;x) - G(-1/y,0,0;x)\Big] \Bigg\} \end{aligned}$$

$$S_{-2} = \frac{1}{32}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right]\left[\frac{1}{y} - 2 + y\right] H(0;x)$$

$$S_{-1} = \frac{1}{64}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right]\left\{\left[\frac{1}{y} - 2 + y\right]\Big[\zeta(2) - 4H(0;x) - H(0,0;x) + 2H(-1,0;x)\Big] - \left[\frac{1}{y} - y\right] H(0;y)H(0;x)\right\}$$
$$+ \frac{1}{32}\left[1 - \frac{2}{(1-x)}\right] H(0;x)$$

$$S_0 = \frac{1}{64}\left\{2\left[2 - \frac{1}{(1-x)}\right]\zeta(2) - 2\left[\frac{1}{(1-y)} - \frac{1}{(1+y)}\right]\Big[\zeta(2)H(0;y) + H(0,0,0;y)\Big] + H(0,0;y) - 5H(0;x) + 2H(-1,0;x)\right.$$
$$+ \frac{2}{(1-x)}\Big[5H(0;x) + H(0,0;x) - 2H(-1,0;x)\Big] + \left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right]\left.\Big[(\zeta(2) - 2H(0,0;y))H(0;x) + H(0,0,0;x)\Big]\right\}$$
$$- \frac{1}{128}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right]\left\{\left[\frac{1}{y} - 2 + y\right]\Big[4\zeta(2) + 2\zeta(3) - 2\zeta(2)H(-1;x) - 2(7 - H(0,0;y))H(0;x) - 4H(0,0;x)\right.$$
$$+8H(-1,0;x) - 2H(0,0,0;x) + 2H(-1,0,0;x) + 2H(0,-1,0;x) - 4H(-1,-1,0;x)\Big] + \left[\frac{1}{y} - y\right]\Big[4\zeta(2)H(0;y) + H(0,0,0;y)$$
$$-2\left(2H(0;y) - \frac{1}{2}\zeta(2) + 2H(0,0;y) + H(1,0;y) - 3H(-1,0;y)\right)H(0;x) + (3\zeta(2) + H(0,0;y))(G(-y;x) - G(-1/y;x))$$
$$\left.+2H(0;y)H(-1,0;x) - H(0;y)G(-y,0;x) - H(0;y)G(-1/y,0;x) + G(-y,0,0;x) - G(-1/y,0,0;x)\Big]\right\}$$

Differential Cross-Section:

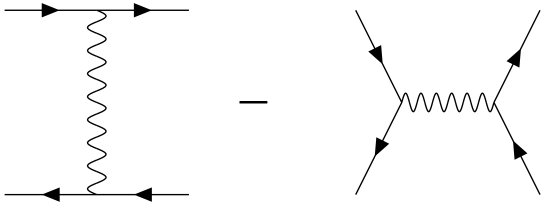
Virtual Contribution

and

Real Soft-Photon Emission

The Differential Cross-Section can be expanded in series of α/π as follows:

$$\frac{d\sigma(s,t,m^2)}{d\Omega} = \frac{d\sigma_0(s,t,m^2)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1(s,t,m^2)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)^2\frac{d\sigma_2(s,t,m^2)}{d\Omega} + \mathcal{O}\left(\left(\frac{\alpha}{\pi}\right)^3\right)$$

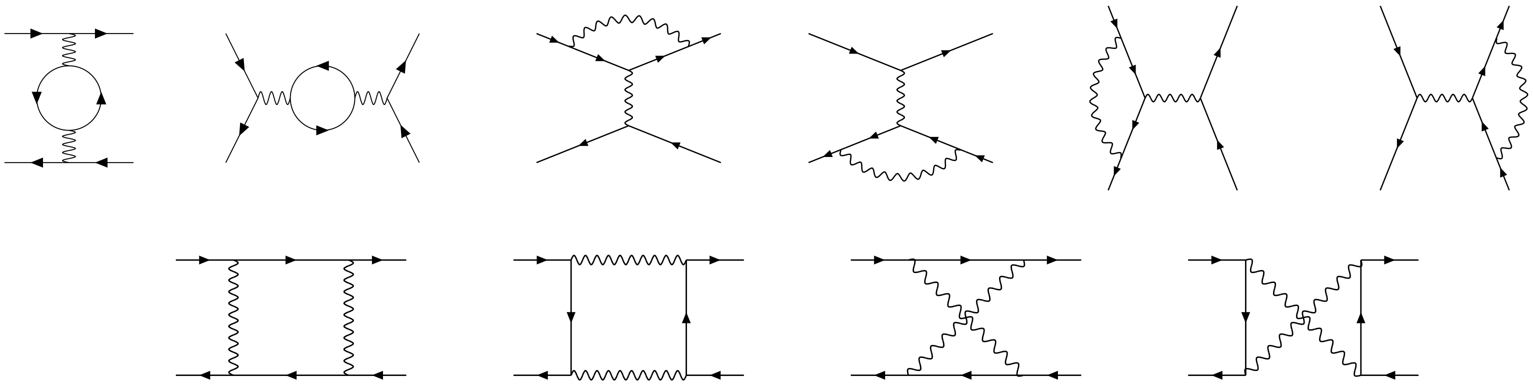
The Tree-Level: $\mathcal{M}_0 =$ 

Performing the traces over the Dirac indices of $||\mathcal{M}_0||^2$ in D dimensions we have

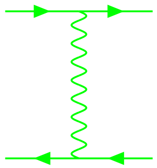
$$\frac{d\sigma_0(s,t,m^2)}{d\Omega} = \frac{\alpha^2}{s}\Bigg\{ \frac{1}{s^2}\left[st + \frac{s^2}{2} + (t-2m^2)^2\right] + \frac{1}{t^2}\left[st + \frac{t^2}{2} + (s-2m^2)^2\right]$$
$$+\frac{1}{st}\left[(s+t)^2 - 4m^4\right]$$
$$+(D-4)\left\{\frac{1}{s^2}\left[\frac{s^2}{4}\right] + \frac{1}{t^2}\left[\frac{t^2}{4}\right] + \frac{1}{st}\left[\frac{1}{2}(s+t)^2 - \frac{1}{2}st - m^2(s+t)\right]\right\}$$
$$+(D-4)^2\left\{\frac{1}{st}\left[-\frac{st}{4}\right]\right\} \Bigg\}$$

$$\frac{d\sigma(s,t,m^2)}{d\Omega} = \frac{d\sigma_0(s,t,m^2)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1(s,t,m^2)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)^2\frac{d\sigma_2(s,t,m^2)}{d\Omega} + \mathcal{O}\left(\left(\frac{\alpha}{\pi}\right)^3\right)$$

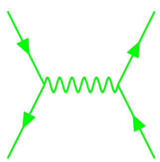
At one loop the following virtual diagrams contribute:



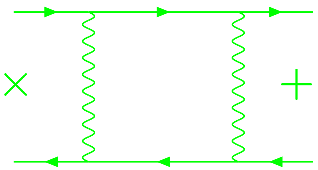
$$\left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1^V(s,t,m^2)}{d\Omega} = \frac{s}{16}\sum_{\text{spin}}\Bigg\{\Bigg($$



$$-$$



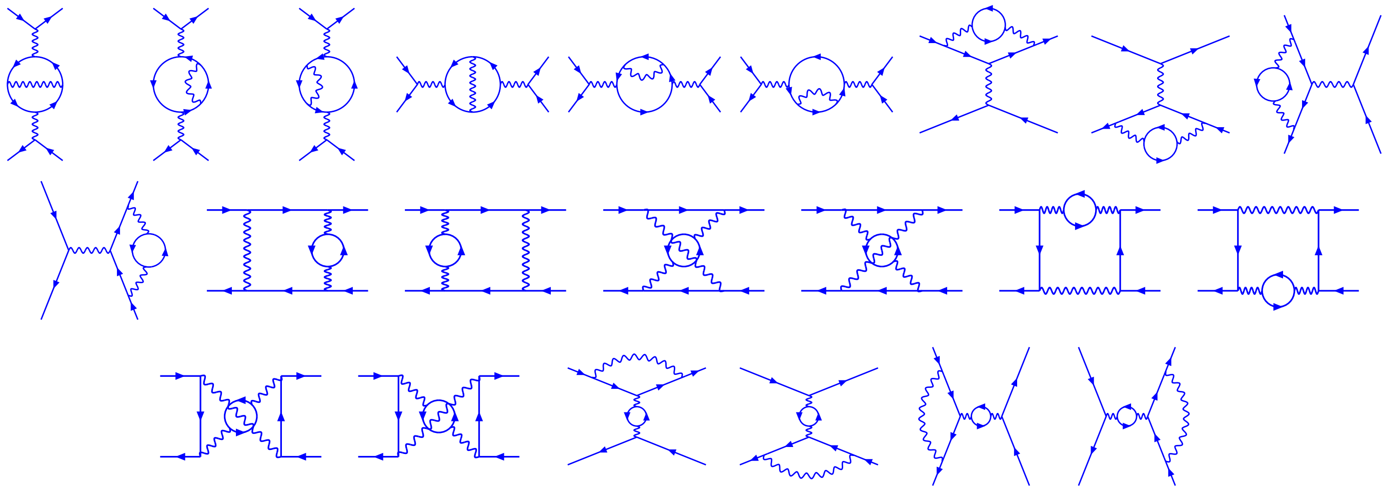
$$\Bigg)^* \times$$



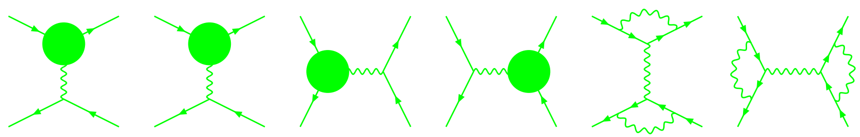
$$+\,\text{c.c.} + \cdots\Bigg\}$$

The $\mathcal{O}(\alpha^4)$ contributions that we consider come from the following Two-Loop diagrams:

$N_F = 1$



Vertices

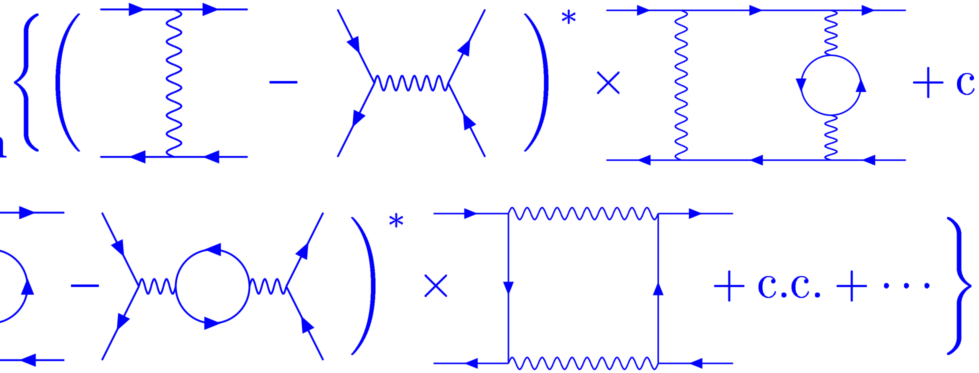


as well as from the product of one-loop diagrams.

$$\frac{d\sigma(s,t,m^2)}{d\Omega} = \frac{d\sigma_0(s,t,m^2)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1(s,t,m^2)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)^2\frac{d\sigma_2(s,t,m^2)}{d\Omega} + \mathcal{O}\left(\left(\frac{\alpha}{\pi}\right)^3\right)$$

where

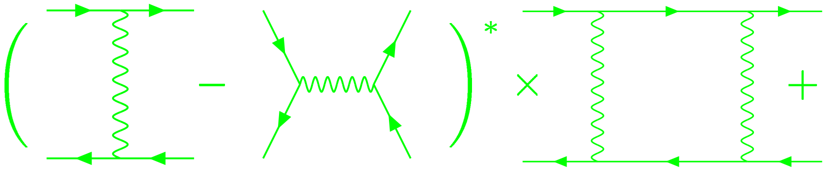
$$\left(\frac{\alpha}{\pi}\right)^2\frac{d\sigma_2^V(s,t,m^2)}{d\Omega} = \frac{s}{16}\sum_{\text{spin}}$$



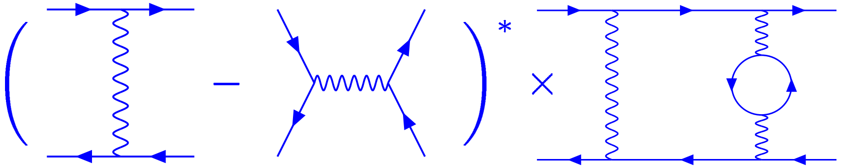
$$\frac{d\sigma(s,t,m^2)}{d\Omega} = \frac{d\sigma_0(s,t,m^2)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1(s,t,m^2)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)^2\frac{d\sigma_2(s,t,m^2)}{d\Omega} + \mathcal{O}\left(\left(\frac{\alpha}{\pi}\right)^3\right)$$

$$\frac{d\sigma_0(s,t,m^2)}{d\Omega} = \frac{\alpha^2}{s}\left\{\frac{1}{s^2}\left[st+\frac{s^2}{2}+(t-2m^2)^2\right]+\frac{1}{t^2}\left[st+\frac{t^2}{2}+(s-2m^2)^2\right] + \frac{1}{st}\left[(s+t)^2-4m^4\right]\right\}$$

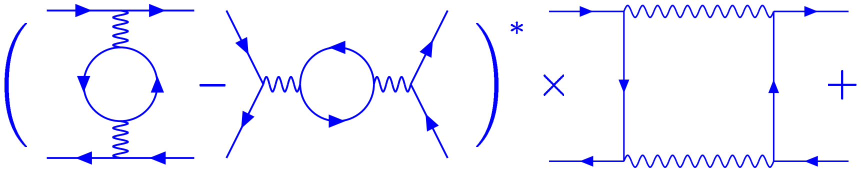
$$\left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1^V(s,t,m^2)}{d\Omega} = \frac{s}{16}\sum_{\text{spin}}\Big\{ \Big(\;\cdots\; - \;\cdots\; \Big)^* \times \;\cdots\; + \text{c.c.} + \cdots \Big\}$$



$$\left(\frac{\alpha}{\pi}\right)^2\frac{d\sigma_2^V(s,t,m^2)}{d\Omega} = \frac{s}{16}\sum_{\text{spin}}\Big\{ \Big(\;\cdots\; - \;\cdots\; \Big)^* \times \;\cdots\; + \text{c.c.}$$



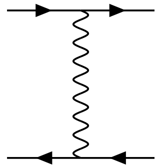
$$+ \Big(\;\cdots\; - \;\cdots\; \Big)^* \times \;\cdots\; + \text{c.c.} + \cdots \Big\}$$



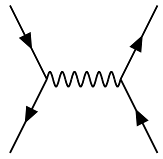
One-Loop

As an example consider

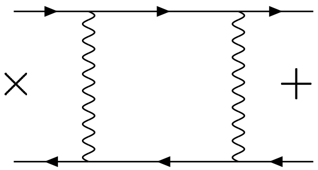
$$\left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1^V(s,t,m^2)}{d\Omega} = \frac{s}{16}\sum_{\text{spin}}\Bigg\{\Bigg($$



$$-$$



$$\Bigg)^* \times$$



$$+\,\text{c.c.}\Bigg\}$$

$$= \frac{\alpha^3}{4\pi s}\left[\frac{m^2}{s}\text{Re}B_1^{(1l)}(s,t) + \frac{m^2}{t}\text{Re}B_2^{(1l)}(s,t)\right]$$

where

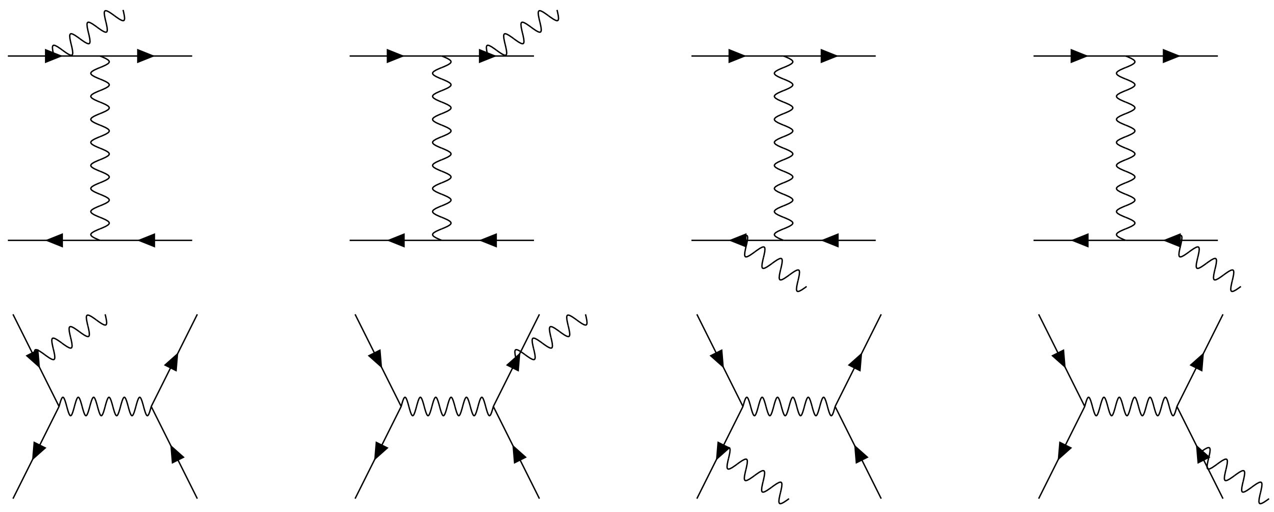
$$B_i^{(1l)}(s,t) = \frac{1}{(D-4)}B_i^{(1l,-1)}(s,t) + B_i^{(1l,0)}(s,t) + \mathcal{O}\Big((D-4)\Big)$$

and the IR pole is

$$B_1^{(1l,-1)}(-P^2,-Q^2) = \Bigg(-48 - \frac{8}{x^2(1-y)^2} + \frac{8}{x^2(1-y)} + \frac{32}{x(1-y)^2} - \frac{32}{x(1-y)} - \frac{16}{x} - \frac{32x}{(1-y)^2} + \frac{32x}{(1-y)} + 16x + \frac{8x^2}{(1-y)^2}$$
$$- \frac{8x^2}{(1-y)} - \frac{8}{y(1+x)} - \frac{8}{y(1-x)} + \frac{8}{y} - \frac{8y}{(1+x)} - \frac{8y}{(1-x)} + 8y - \frac{96}{(1+x)(1-y)^2} + \frac{96}{(1+x)(1-y)} + \frac{80}{(1+x)}$$
$$+ \frac{32}{(1-x)(1-y)^2} - \frac{32}{(1-x)(1-y)} + \frac{16}{(1-x)} + \frac{32}{(1-y)^2} - \frac{32}{(1-y)}\Bigg)H(0;x)$$

In order to get a finite differential cross-section we have to add the real soft-photon emission, which cancels the IR poles coming from the virtual cross-section.

At $\mathcal{O}(\alpha^3)$ the diagrams contributing are the following



Due to the iconalization of the vertex, the real emission contribution at $\mathcal{O}(\alpha^3)$ is given by

$$\left(\frac{\alpha}{\pi}\right) \frac{d\sigma_1^S(s,t,m^2)}{d\Omega} = \left(\frac{\alpha}{\pi}\right) \frac{d\sigma_0^D(s,t,m^2)}{d\Omega} \sum_{i,j=1}^{4} J_{ij}$$

with

$$J_{ij} = \epsilon_i \epsilon_j \; (p_i \cdot p_j) \; I_{ij},$$

$$\epsilon_i = +1 \text{ for } i = 1,4 \text{ and } \epsilon_i = -1 \text{ for } i = 2,3$$

and

$$I_{ij} = \frac{1}{\Gamma\left(3 - \frac{D}{2}\right) \pi^{(D-4)/2}} \frac{m^{D-4}}{4\pi^2} \int^{\omega} \frac{d^D k}{k_0} \frac{1}{(p_i \cdot k)(p_j \cdot k)}$$

- ω is the cut-off on the unobserved photon energy
- because $J_{ij} = J_{ji}$ we have in fact:

$$\left(\frac{\alpha}{\pi}\right) \frac{d\sigma_1^S(s,t,m^2)}{d\Omega} = \left(\frac{\alpha}{\pi}\right) \frac{d\sigma_0^D(s,t,m^2)}{d\Omega} \, 4 \sum_{j=1}^{4} J_{1j}$$

where $d\sigma_0^D(s,t,m^2)/d\Omega$ is the Born cross-section exact in D.

$$I_{1j} = \frac{\rho_{1j}}{2}\left[\left(\frac{1}{D-4} + \frac{1}{2}\ln\left(\frac{\omega^2}{m^2}\right) + \ln 2\right) I_{1j}^{(0)} - \Delta I_{1j}\right] + \mathcal{O}\left(D-4\right)$$

$$\rho_{11} = 1\,, \quad \rho_{12} = \frac{1}{x}\,, \quad \rho_{13} = \frac{1}{y}\,, \quad \rho_{14} = \frac{1}{z}$$

$$x = \frac{\sqrt{s} - \sqrt{s-4m^2}}{\sqrt{s} + \sqrt{s-4m^2}}\,, \quad y = \frac{\sqrt{4m^2-t} - \sqrt{-t}}{\sqrt{4m^2-t} + \sqrt{-t}}\,, \quad z = \frac{\sqrt{4m^2-u} - \sqrt{-u}}{\sqrt{4m^2-u} + \sqrt{-u}}$$

$$I_{1j}^{(0)} = \frac{2}{m^2}\int_0^1 dr \left[1 - 2r\left(1 + \rho_{1j}\frac{p_1 \cdot p_j}{m^2}\right)\right]^{-1}$$

$$\Delta I_{1j} = \int_0^1 dr \frac{1}{(P_{1j})^2}\frac{(P_{1j})_0}{|\vec{P_{1j}}|}\ln\frac{(P_{1j})_0 - |\vec{P_{1j}}|}{(P_{1j})_0 + |\vec{P_{1j}}|}$$

$$P_{1j}^{\mu} = p_j^{\mu} + r\left(\rho_{1j}p_1^{\mu} - p_j^{\mu}\right)$$

$$I_{11}^{(0)} = \frac{2}{m^2}\,, \qquad I_{12}^{(0)} = -\frac{4}{m^2}\frac{x^2}{1-x^2}\ln x$$

$$I_{13}^{(0)} = -\frac{4}{m^2}\frac{y^2}{1-y^2}\ln y\,, \qquad I_{14}^{(0)} = -\frac{4}{m^2}\frac{z^2}{1-z^2}\ln z$$

$$\Delta I_{11} = -\frac{1}{m^2}\frac{1+x}{1-x}\ln x$$

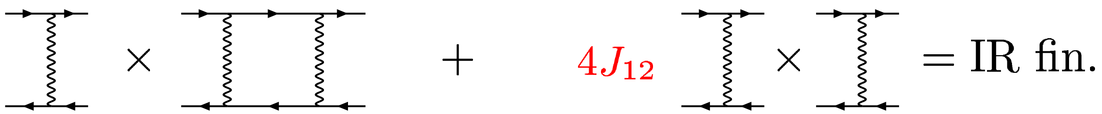
$$\Delta I_{1l} = -\frac{2}{m^2(\rho_{1l}^2-1)}\Big[\mathrm{Li}_2(a_l^{(1)}) + \mathrm{Li}_2(a_l^{(2)}) - \mathrm{Li}_2(a_l^{(3)}) - \mathrm{Li}_2(a_l^{(4)})\Big]$$

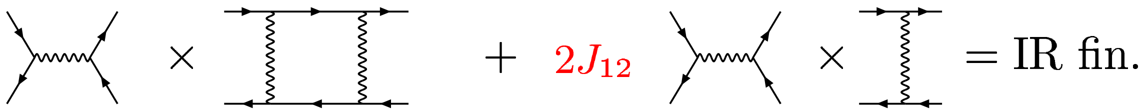
$$a_l^{(1)} = \frac{1-x\rho_{1l}}{1+\rho_{1l}}\,, \quad a_l^{(2)} = \frac{x-\rho_{1l}}{x(1+\rho_{1l})}\,, \quad a_l^{(3)} = \frac{\rho_{1l}-x}{1+\rho_{1l}}\,, \quad a_l^{(4)} = -\frac{1-x\rho_{1l}}{x(1+\rho_{1l})}$$

Finally, the contribution UV-renormalized and IR finite at $\mathcal{O}(\alpha^3)$ is given by

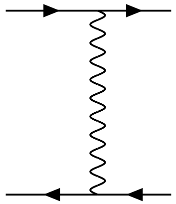
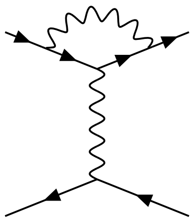
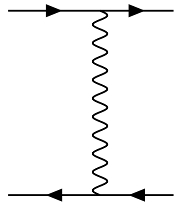
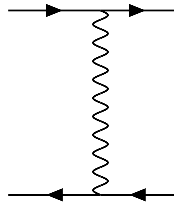
$$
\begin{aligned}
\left(\frac{\alpha}{\pi}\right) \frac{d\sigma_1(s,t,m^2)}{d\Omega} &= \left(\frac{\alpha}{\pi}\right) \left(\frac{d\sigma_1^V(s,t,m^2)}{d\Omega} + \frac{d\sigma_1^S(s,t,m^2)}{d\Omega} \right) \\
&= \left(\frac{\alpha}{\pi}\right) \left(\frac{d\sigma_1^V(s,t,m^2)}{d\Omega} + \frac{d\sigma_0^D(s,t,m^2)}{d\Omega} \, 4 \sum_{j=1}^{4} J_{1j} \right)
\end{aligned}
$$

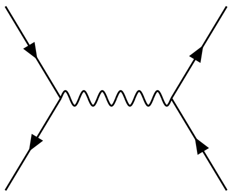
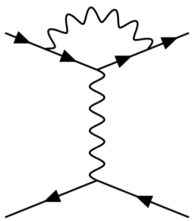
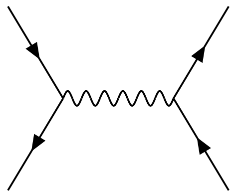
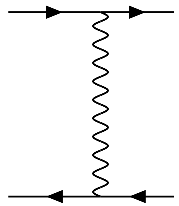
In particular, for our example we find

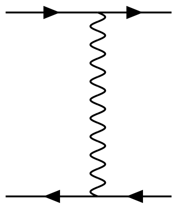
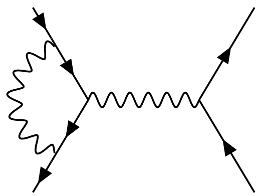
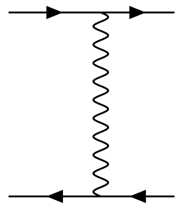
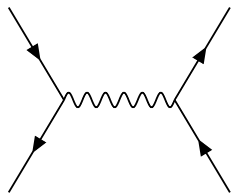


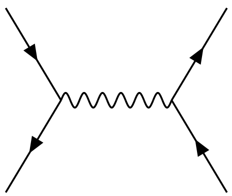
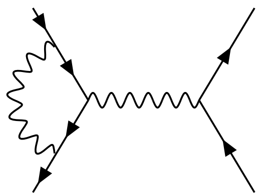
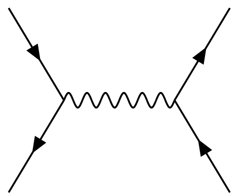
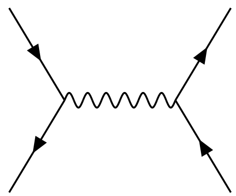


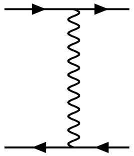
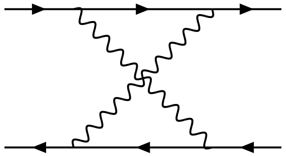
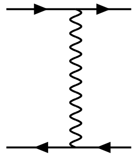
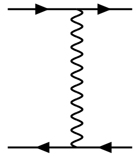
Note the fact that we need the $\mathcal{O}(D-4)$ of the Born cross-section in order to get its correct finite part!

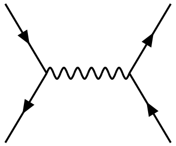
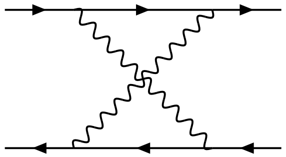
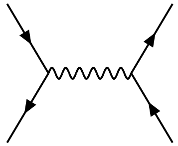
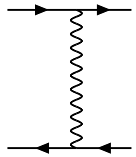
 $\times$  $+ \quad 2(J_{11} + J_{13})$  $\times$  $= \text{IR fin.}$

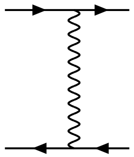
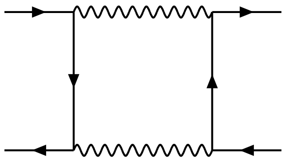
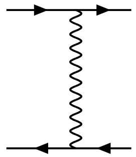
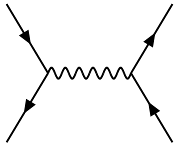
 $\times$  $+ \quad (J_{11} + J_{13})$  $\times$  $= \text{IR fin.}$

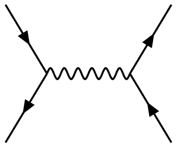
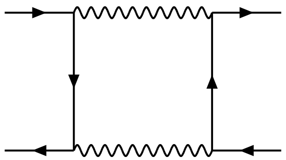
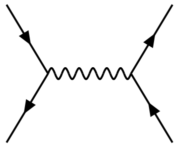
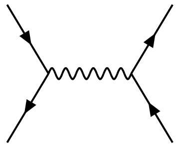
 $\times$  $+ \quad (J_{11} + J_{12})$  $\times$  $= \text{IR fin.}$

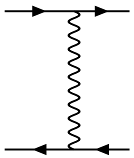
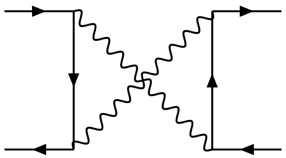
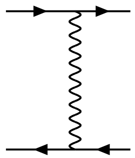
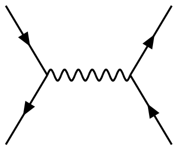
 $\times$  $+ \quad 2(J_{11} + J_{12})$  $\times$  $= \text{IR fin.}$

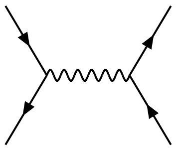
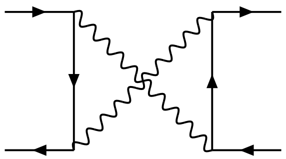
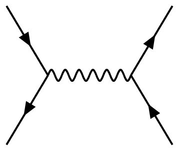
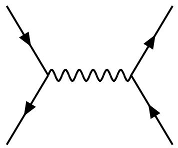
 $\times$  $+$ $4J_{14}$  $\times$  $=$ IR fin.

 $\times$  $+$ $2J_{14}$  $\times$  $=$ IR fin.

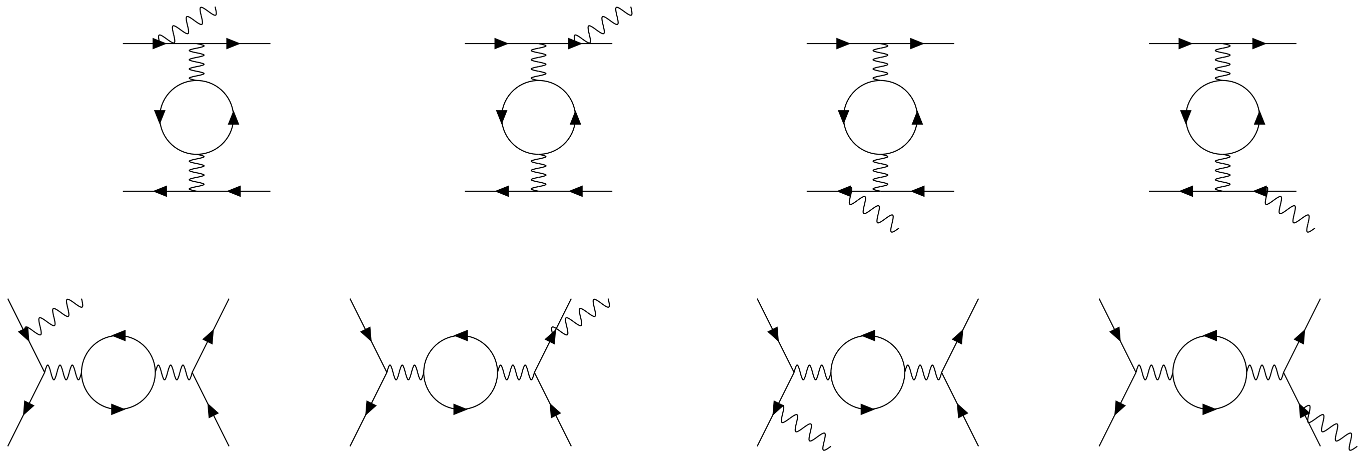
 $\times$  $+$ $2J_{13}$  $\times$  $=$ IR fin.

 $\times$  $+$ $4J_{13}$  $\times$  $=$ IR fin.

 $\times$  $+$ $2J_{14}$  $\times$  $=$ IR fin.

 $\times$  $+$ $4J_{14}$  $\times$  $=$ IR fin.

Real photon emission for the $\mathcal{O}(\alpha^4 N_F = 1)$



$$\left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma_2^S(s,t,m^2)}{d\Omega}\Bigg|_{N_F=1} = \left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma_1^D(s,t,m^2)}{d\Omega}\Bigg|_{N_F=1} 4 \sum_{j=1}^{4} J_{1j}$$

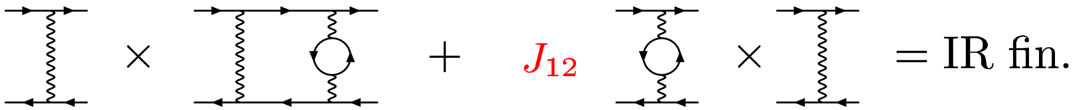
- $d\sigma_1^D(s,t,m^2)/d\Omega|_{N_F=1}$ is the one-loop $N_F = 1$ cross-section as a series in $(D-4)$.
- Note the fact that we need the $\mathcal{O}(D-4)$ of the $N_F = 1$ cross-section in order to get the correct finite part of the real emission, but again we DO NOT need the $\mathcal{O}(D-4)$ of the integrals J_{1j}, because $d\sigma_1^D(s,t,m^2)/d\Omega|_{N_F=1}$ is finite!

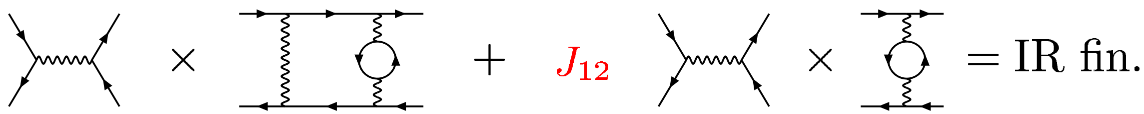
Finally, the contribution UV-renormalized and IR finite at $\mathcal{O}(\alpha^4 N_F = 1)$ is given by

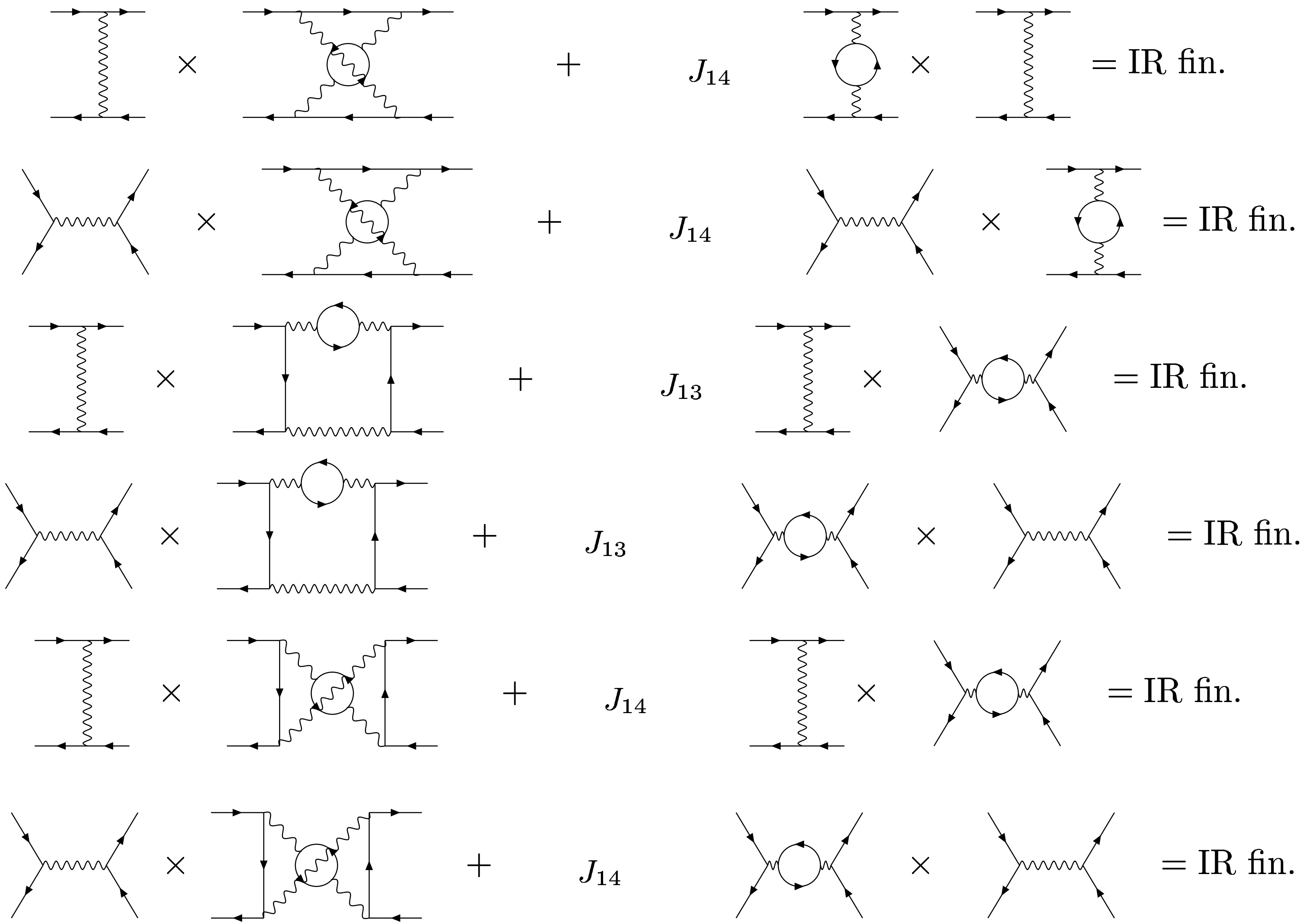
$$\left(\frac{\alpha}{\pi}\right) \frac{d\sigma_2(s,t,m^2)}{d\Omega}\bigg|_{N_F=1} = \left(\frac{\alpha}{\pi}\right) \left(\frac{d\sigma_2^V(s,t,m^2)}{d\Omega}\bigg|_{N_F=1} + \frac{d\sigma_1^S(s,t,m^2)}{d\Omega}\bigg|_{N_F=1} \right)$$

$$= \left(\frac{\alpha}{\pi}\right) \left(\frac{d\sigma_2^V(s,t,m^2)}{d\Omega}\bigg|_{N_F=1} + \frac{d\sigma_1^D(s,t,m^2)}{d\Omega}\bigg|_{N_F=1} 4 \sum_{j=1}^{4} J_{1j} \right)$$

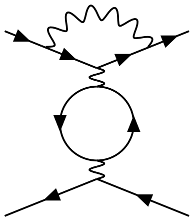
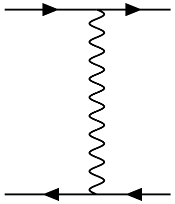
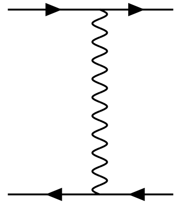
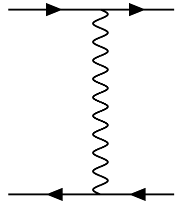
In particular, for a two-loop box we find

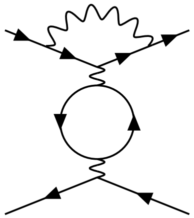
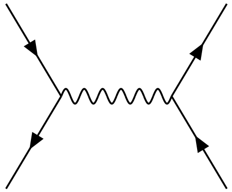
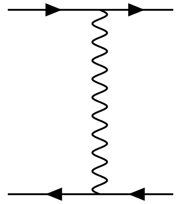
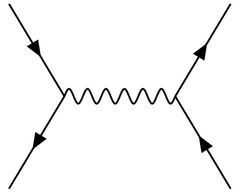


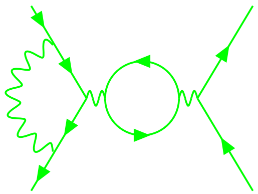
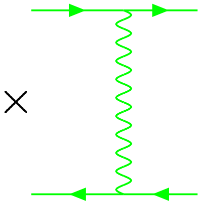
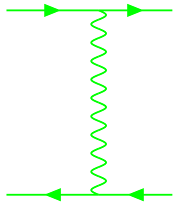
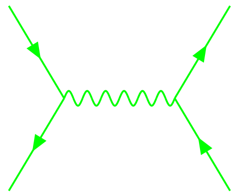


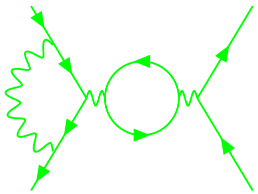
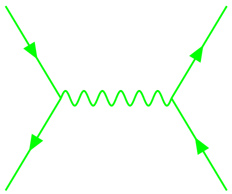
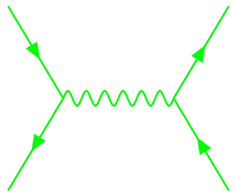
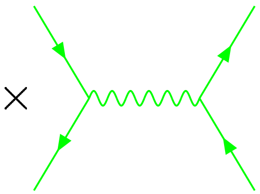

× + J_{14} × = IR fin.
× + J_{14} × = IR fin.
× + J_{13} × = IR fin.
× + J_{13} × = IR fin.
× + J_{14} × = IR fin.
× + J_{14} × = IR fin.

Reducible Diagrams $\mathcal{O}(\alpha^4 N_F = 1)$

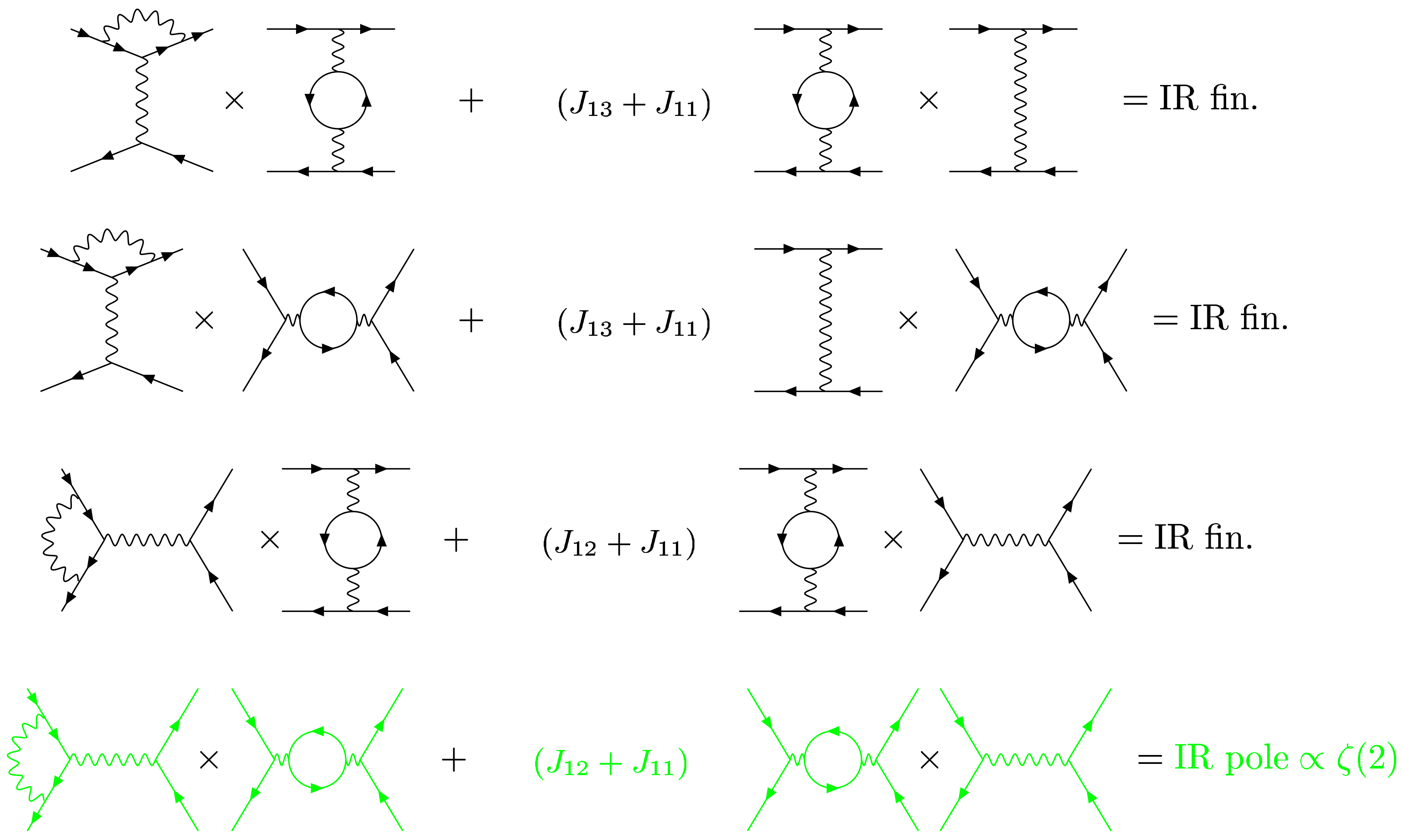
 $\times$  $+ \quad 2\,(J_{13} + J_{11})$  $\times$  $=$ IR fin.

 $\times$  $+ \quad (J_{13} + J_{11})$  $\times$  $=$ IR fin.

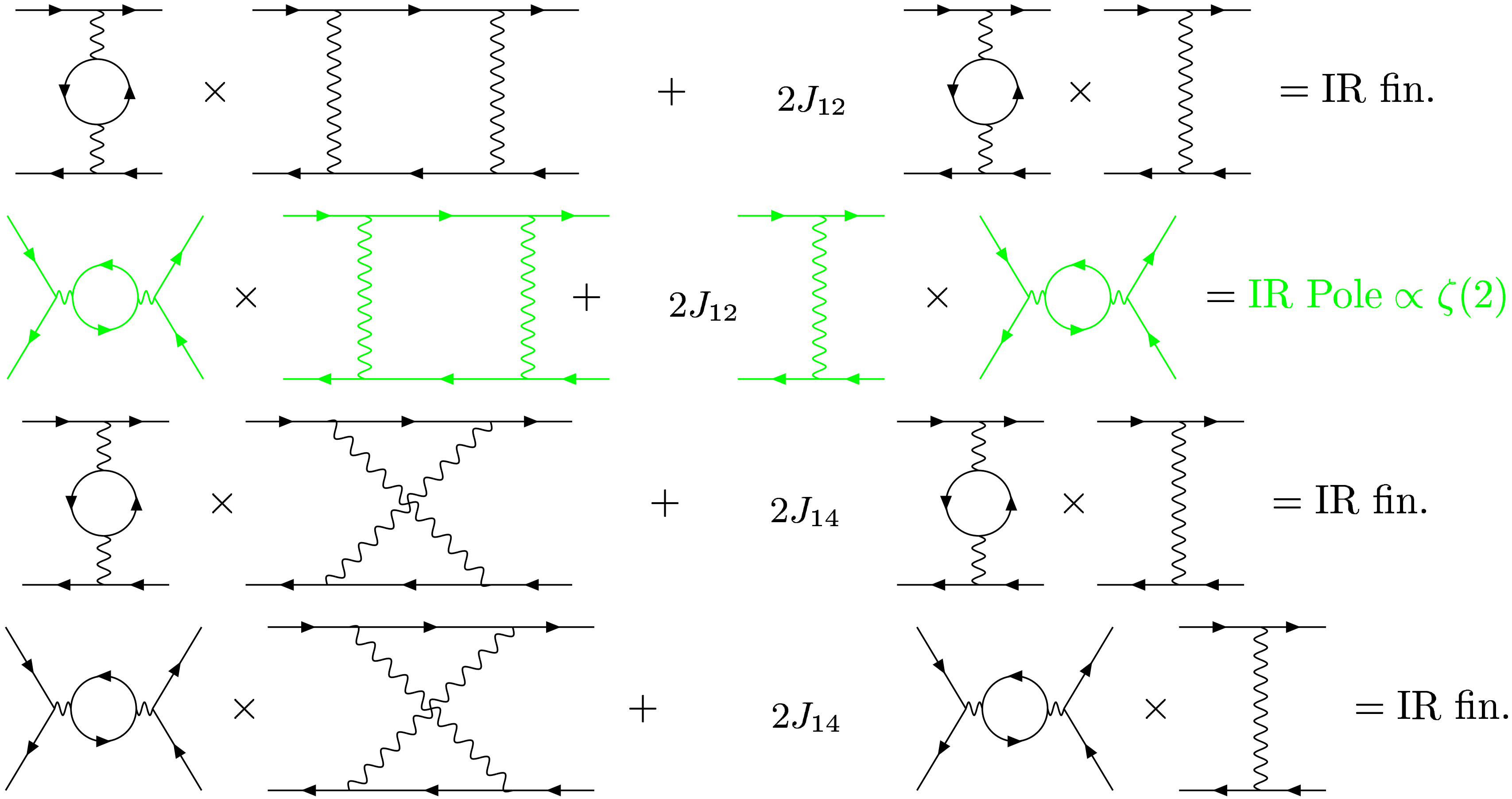
 $\times$  $+ \quad (J_{12} + J_{11})$  $\times$  $=$ IR pole $\propto \zeta(2)$

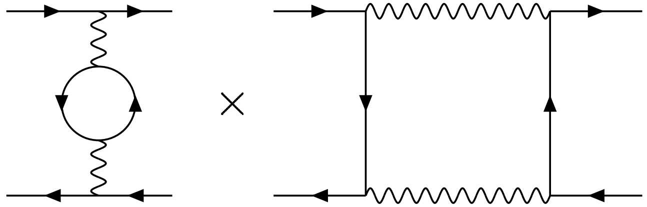
 $\times$  $+ \quad 2\,(J_{12} + J_{11})$  $\times$  $=$ IR pole $\propto \zeta(2)$

One-Loop Vertices times One-Loop $\mathcal{O}(\alpha^4 N_F = 1)$

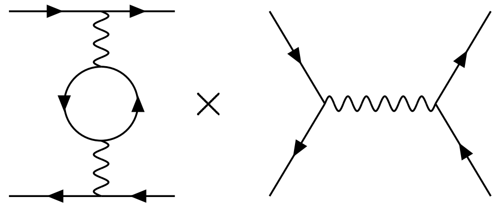


One-Loop boxes times One-loop $\mathcal{O}(\alpha^4 N_F = 1)$

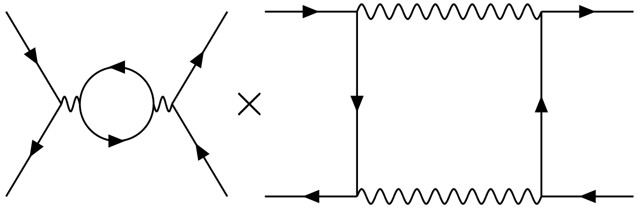




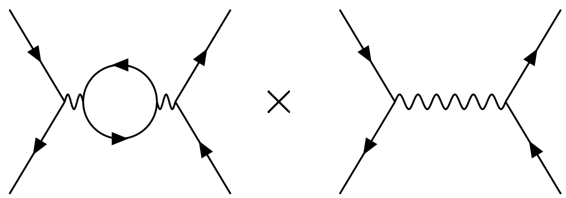
$$+ \quad 2J_{13}$$



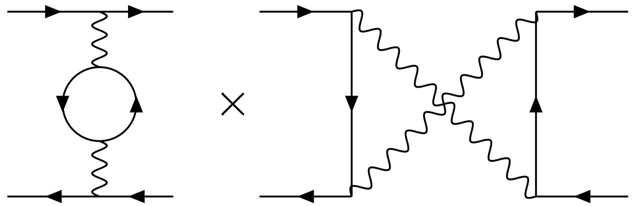
$$= \text{IR fin.}$$



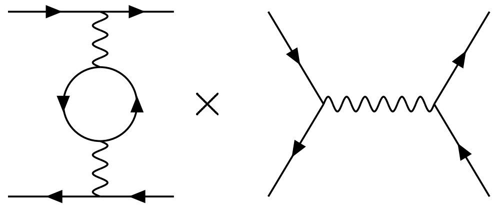
$$+ \quad 2J_{13}$$



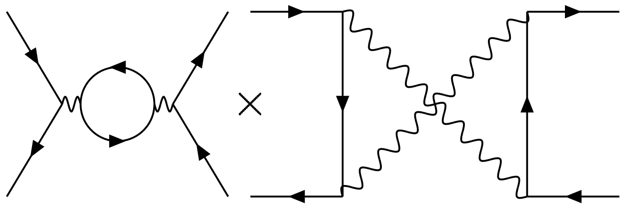
$$= \text{IR fin.}$$



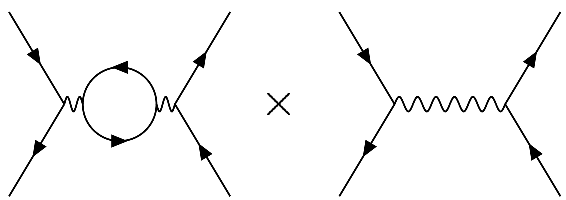
$$+ \quad 2J_{14}$$



$$= \text{IR fin.}$$

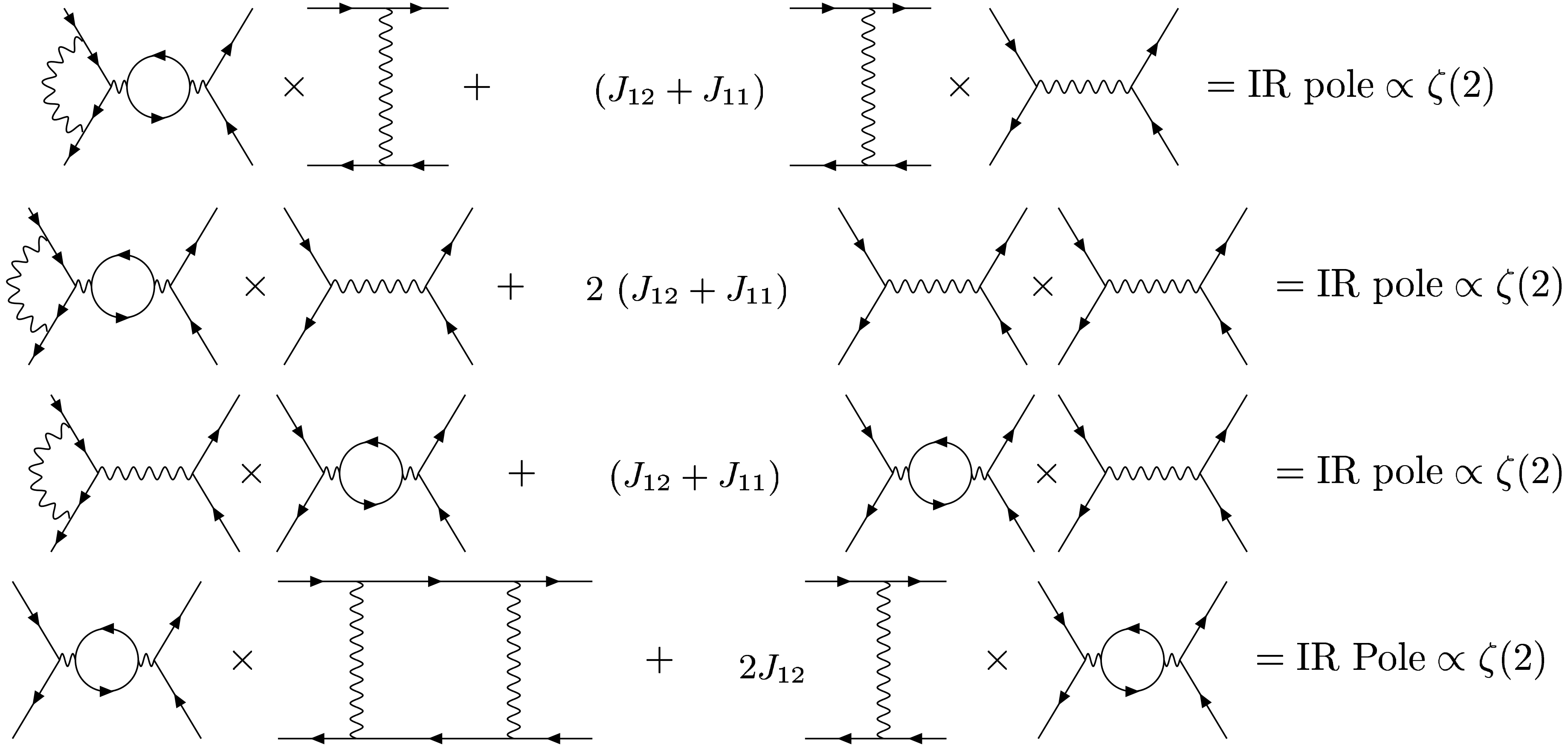


$$+ \quad 2J_{14}$$



$$= \text{IR fin.}$$

Single IR Poles proportional to $\zeta(2)$

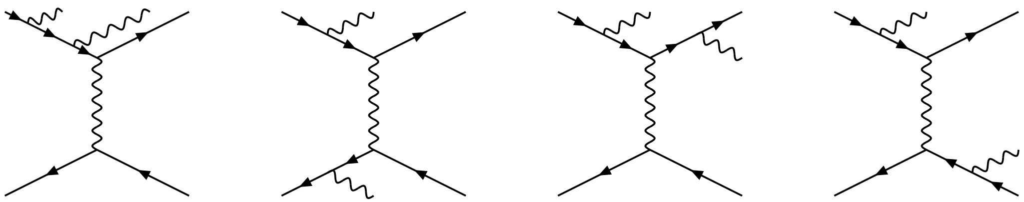


These poles cancel in the final sum!

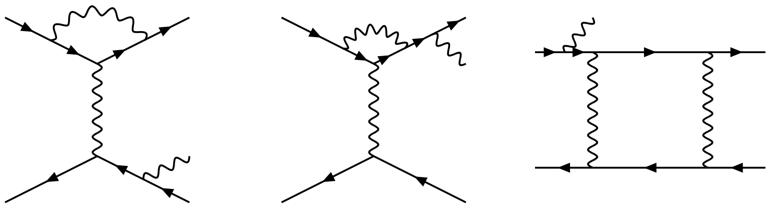
Real photon emission at $\mathcal{O}(\alpha^4)$

The inclusion of the two-loop vertex corrections requires the following real soft-emission diagrams

Tree-Level with the emission of two photons



One-Loop diagrams with the emission of one photon



....

Since the soft-photon corrections in QED exponentiate, the contribution due to the emission of two photons from the tree-level and the one due to the emission of one photon from the one-loop amplitude are given respectively by:

$$\left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma_2^{S,2-ph}(s,t,m^2)}{d\Omega} = \frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma_0^{D}(s,t,m^2)}{d\Omega} \left(4 \sum_{j=1}^{4} J_{1j}\right)^2$$

$$\left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma_2^{S,1-ph}(s,t,m^2)}{d\Omega} = \left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma_1^{(V,D)}(s,t,m^2)}{d\Omega}\, 4 \sum_{j=1}^{4} J_{1j}$$

Note that:

- the term $(4 \sum_{j=1}^{4} J_{1j})^2$ in the first equation has a double IR pole: therefore also the $\mathcal{O}(D-4)^2$ of $d\sigma_0^D/d\Omega$ is required;
- because $d\sigma_1^{(V,D)}/d\Omega$ is IR-divergent, in principle we need also the $\mathcal{O}(D-4)$ of the integrals J_{1j}, BUT it is possible to prove that actually we DO NOT need it.

Cancellation of $\mathcal{O}(D-4)$ of J_{1j}

To prove the fact that we do not need to know the $\mathcal{O}(D-4)$ of the integrals J_{1j}, consider that

- $d\sigma_1^{(V,D)}/d\Omega$ includes the contribution of the vertices and the boxes

$$\frac{d\sigma_1^{(V,D)}(s,t,m^2)}{d\Omega} = \left.\frac{d\sigma_1^{(V,D)}(s,t,m^2)}{d\Omega}\right|_{(1l,V)} + \left.\frac{d\sigma_1^{(V,D)}(s,t,m^2)}{d\Omega}\right|_{(1l,B)}$$

- These two contributions are IR divergent:

$$\left.\frac{d\sigma_1^{V}(s,t,m^2)}{d\Omega}\right|_{(1l,j)} = \frac{1}{(D-4)}\left.\frac{d\sigma_1^{(V,-1)}(s,t,m^2)}{d\Omega}\right|_{(1l,j)} + \left.\frac{d\sigma_1^{(V,0)}(s,t,m^2)}{d\Omega}\right|_{(1l,j)}$$

- If we write $4\sum_{j=1}^{4} J_{1j} = S_{\rm IR}$, we also have:

$$S_{\rm IR} = \frac{S_{\rm IR}^{(-1)}}{(D-4)} + S_{\rm IR}^{(0)} + (D-4)S_{\rm IR}^{(1)} + \mathcal{O}\left((D-4)^2\right)$$

- The cancellation of the IR divergences in the $\mathcal{O}(\alpha^3)$ cross-section guarantees that

$$\left.\frac{d\sigma_1^{(V,-1)}(s,t,m^2)}{d\Omega}\right|_{(1l,B)} + \left.\frac{d\sigma_1^{(V,-1)}(s,t,m^2)}{d\Omega}\right|_{(1l,V)} + \frac{d\sigma_0(s,t,m^2)}{d\Omega} S_{\rm IR}^{(-1)} = 0$$

- From the double emission cross-section we find that the term proportional to $S_{\rm IR}^{(1)}$ is

$$\frac{d\sigma_2^{(S,\texttt{double})}(s,t,m^2)}{d\Omega} \to \frac{d\sigma_0(s,t,m^2)}{d\Omega} S_{\rm IR}^{(-1)} S_{\rm IR}^{(1)}$$

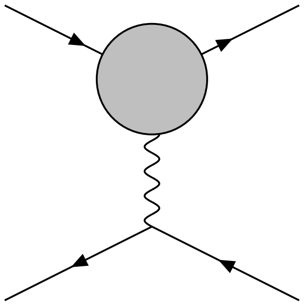
- The term proportional to $S_{\rm IR}^{(1)}$ appearing in the single photon emission cross-sections is

$$\frac{d\sigma_2^{(S,\texttt{single})}(s,t,m^2)}{d\Omega} \to \left[\left.\frac{d\sigma_1^{(V,-1)}(s,t,m^2)}{d\Omega}\right|_{(1l,B)} + \left.\frac{d\sigma_1^{(V,-1)}(s,t,m^2)}{d\Omega}\right|_{(1l,V)} \right] S_{\rm IR}^{(1)}$$

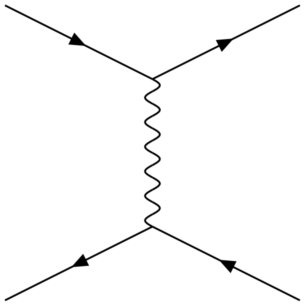
We conclude that the terms proportional to $S_{\rm IR}^{(1)}$ cancel out in the total real-emission cross-section at order $\mathcal{O}(\alpha^4)$!

Diagrammatic IR subtraction

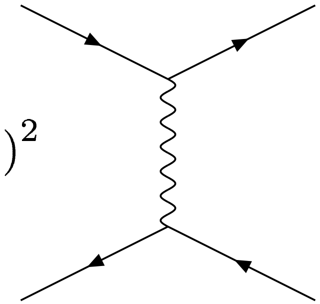
As for the previous contributions, we have particular combinations of diagrams that are IR finite:



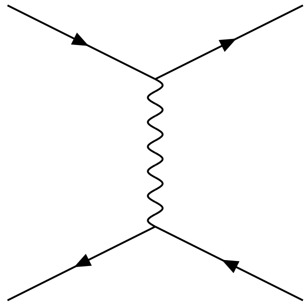
$\times$



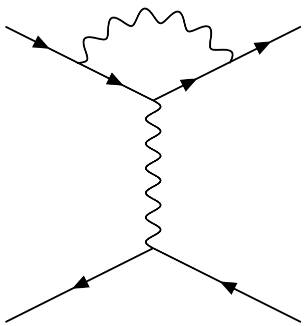
$$+\frac{1}{16}\left(J_{11}+J_{13}\right)^2$$



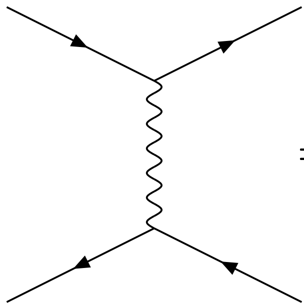
$\times$



$$+\frac{1}{4}\left(J_{11}+J_{13}\right)$$

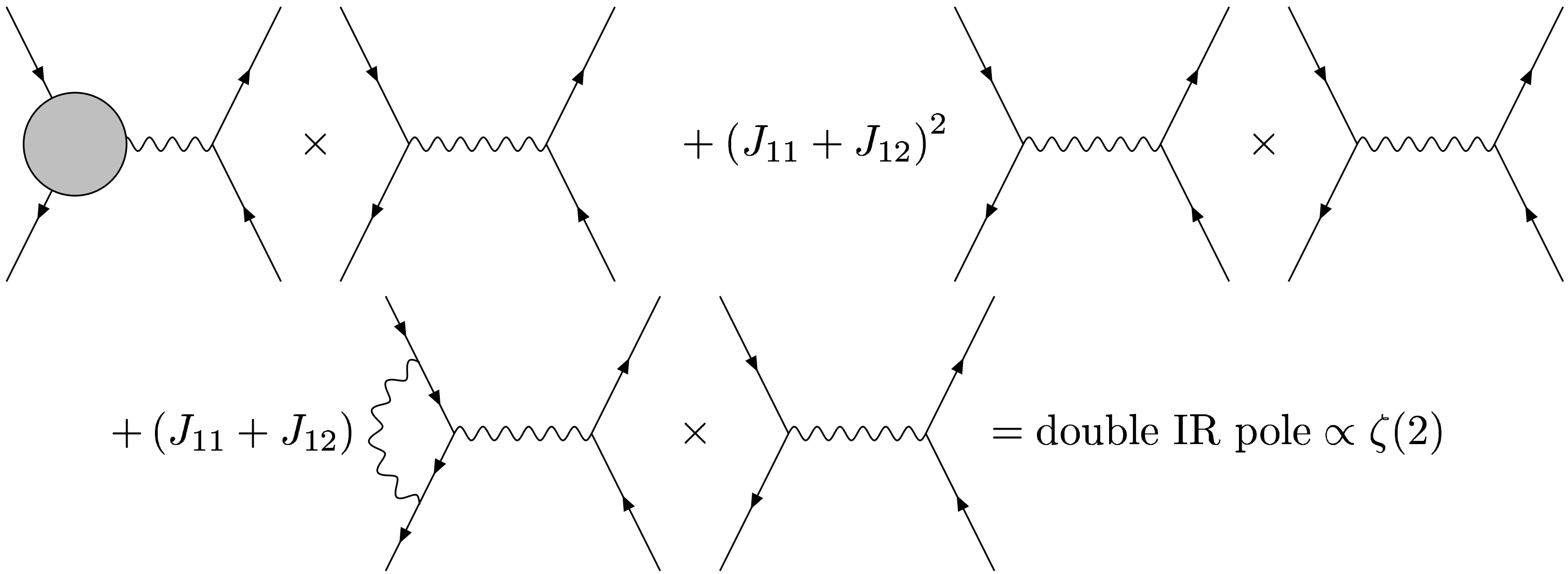


$\times$



$= \text{IR finite}$

We have also residual IR poles proportional to $\zeta(2)$:



These poles need, in order to be canceled, the contributions due to the photonic box diagrams

Summary and Outlook

- The calculation of the Two-Loop $N_F = 1$ and Two-Loop Vertex (unpublished yet) QED corrections to the Bhabha scattering differential cross-section was carried out keeping the electron mass different from zero, by means of the Laporta algorithm and the Differential Equations Method
- As a remark, we find that, from a phenomenological point of view, there is no difference between the exact calculation and the LL approximation (both for the $N_F = 1$ calculation and for the vertices)
- The numerical evaluation of the analytic expressions presented in the talk is implemented in two computer programs, written in Fortran and Mathematica. At the LL approximation, the full set of Two-Loop QED contributions are included ($N_F = 1$ as well as Penin's formulas). The exact results, instead, include everything except the Two-Loop photonic-box corrections
- The Two-Loop photonic-box corrections are still missing